Lookahead Games for History-Deterministic Parity Automata

Rohan Acharya, Marcin Jurdziński, Aditya Prakash

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Deterministic Parity Automata

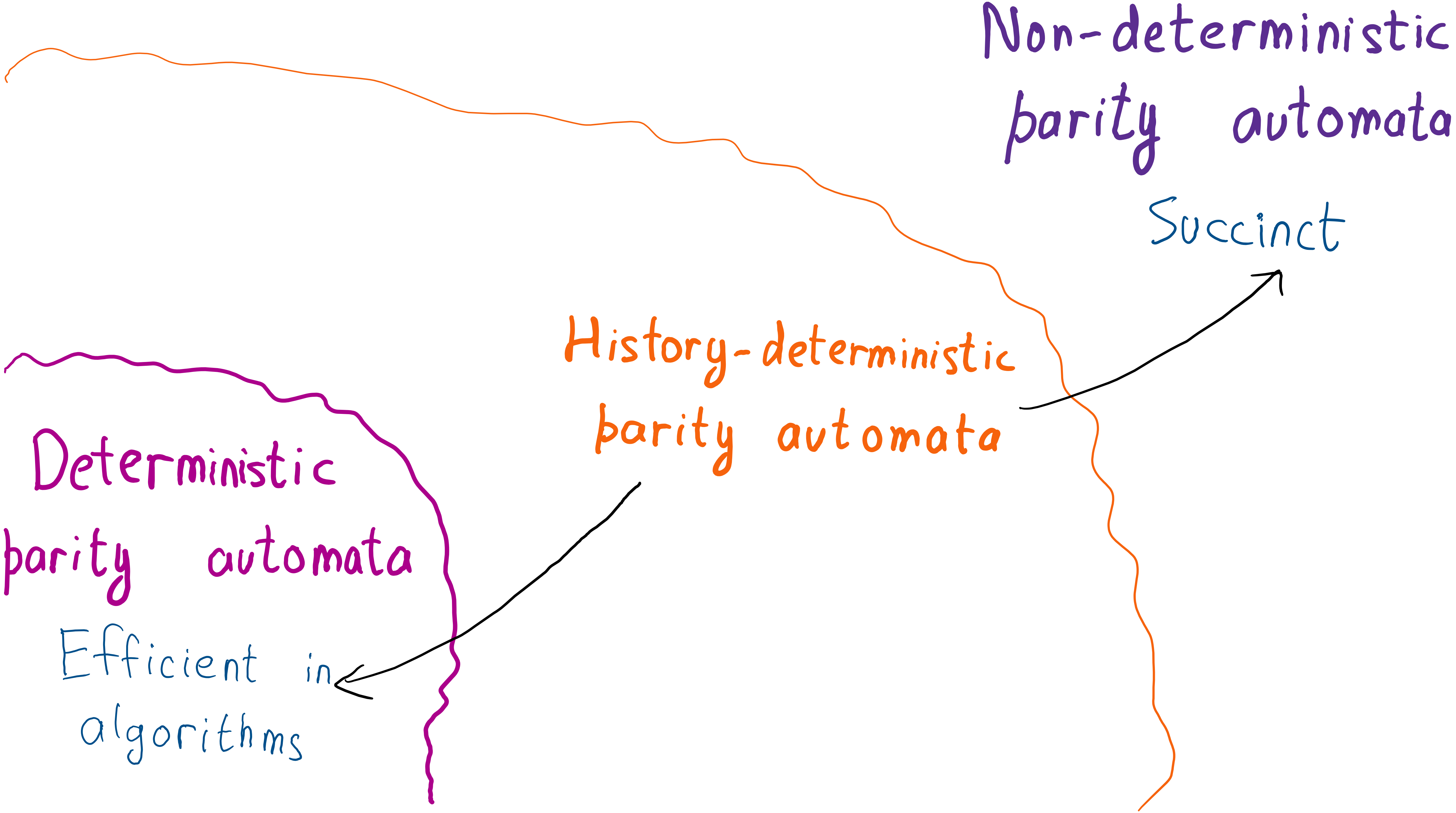
Efficient in algorithms

Non-deterministic parity automata

Succinct

Deterministic parity automata

Efficient in algorithms


Non-deterministic parity automata
Succinct
History-deterministic parity automata
Deterministic parity automata
Efficient in algorithms

Problem: Given: A ← Non-det. parity automaton

Is A HD?

Complexity: OPEN since 2006

* Checking History-Determinism is NP-hard for Parity Automata AP FoSSaCS'24

* Lookahead Games and Efficient Determinisation of HD Büchi Automata
Rohan Acharya, Marcin Jurdziński, AP ICALP'24

I. (Recognising HD Parity Automata)

Parity Condition

3, 1, 2, 1, 2, ...

4, 2, 3, 2, 3, ...

$\rightarrow$ Sequence of natural numbers

Parity Condition

3, 1, 2, 1, 2, ...
4, 2, 3, 2, 3, ... $\Big\}\to$ Sequence of natural numbers

"Highest number occurring ∞ly often is even."

Parity Condition

3, 1, 2, 1, 2, ...
4, 2, 3, 2, 3, ... → Sequence of natural numbers

"Highest number occurring ∞ly often is even."

Parity Condition

3, 1, $\underline{\underline{2}}$, 1, 2, ... ✓

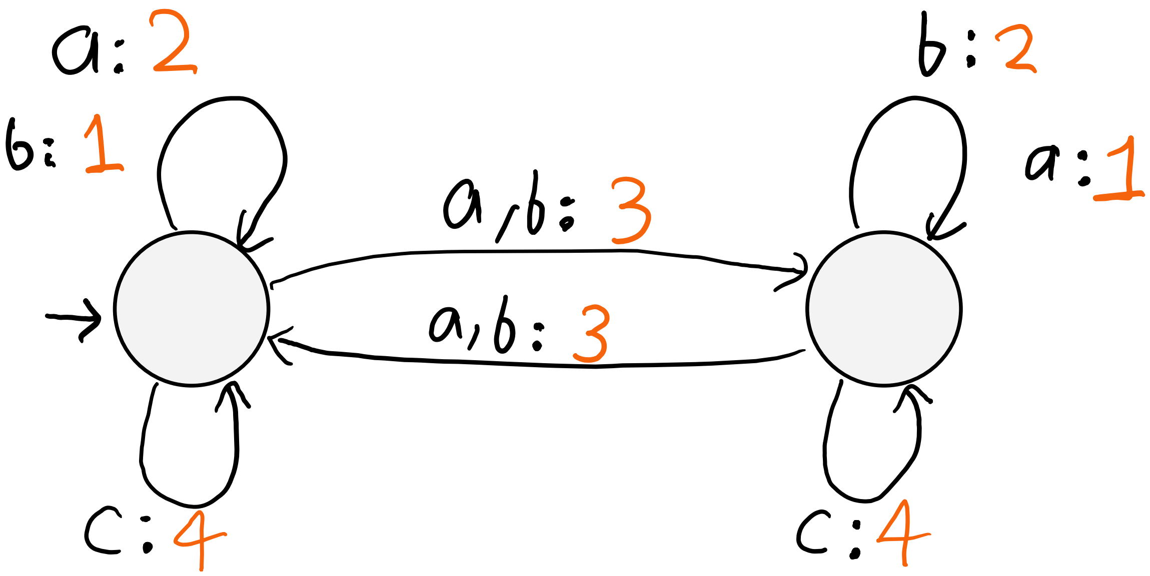
4, 2, $\underline{\underline{3}}$, 2, 3, ... ✗

→ Sequence of natural numbers

"Highest number occurring ∞ly often is even."

Parity Automata

Input: $w \in \{a,b,c\}^{\mathbb{N}}$



Parity Automata

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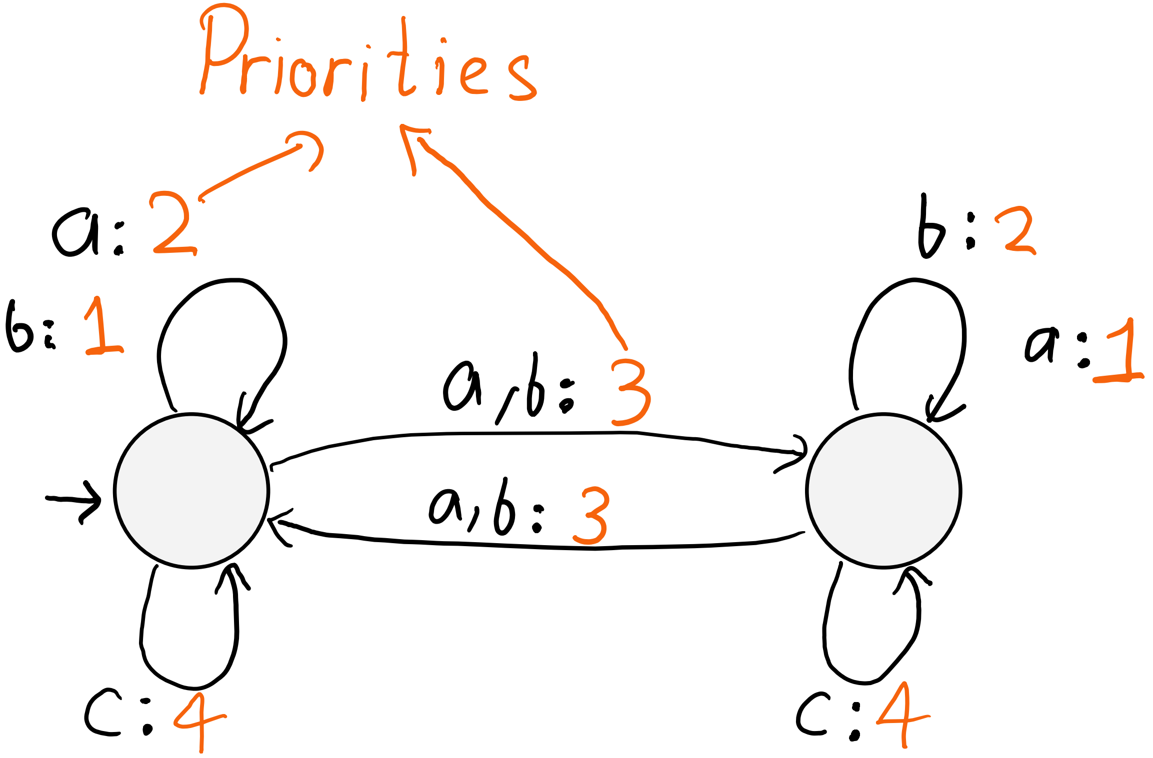
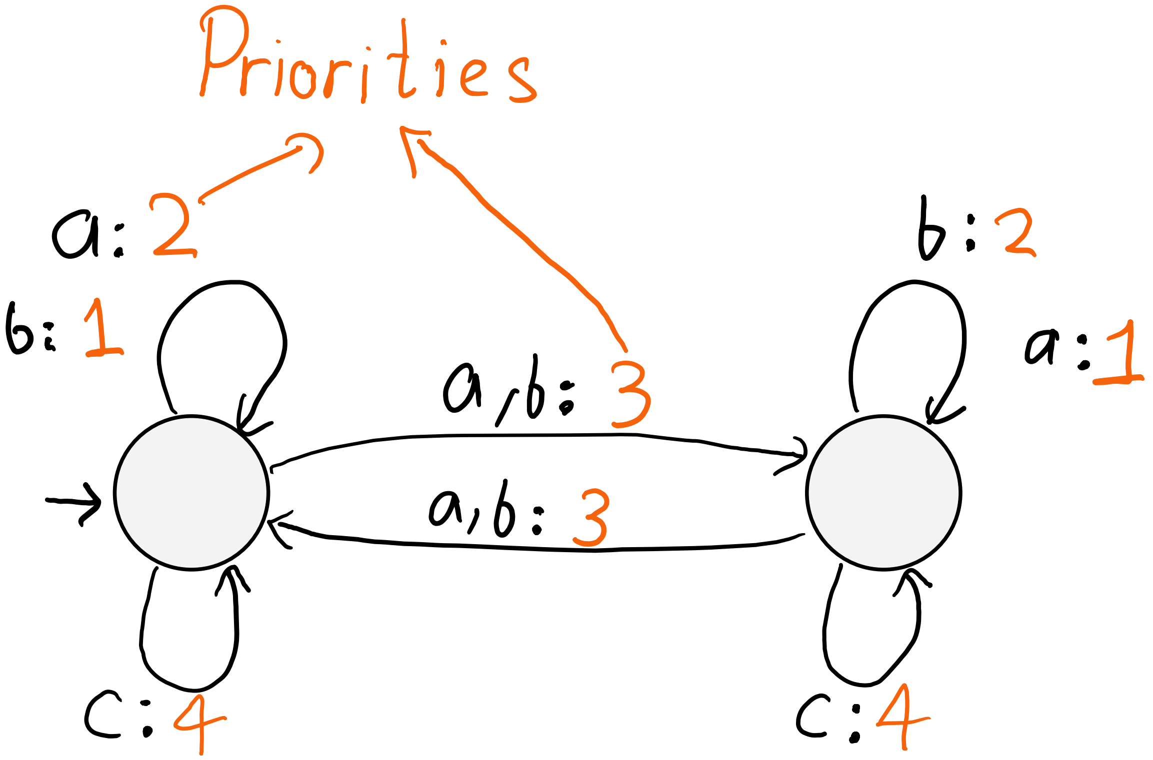


Accepting run:

Sequence of priorities satisfies the parity condition.

Parity Automata

Input: $w \in \{a,b,c\}^{\mathbb{N}}$

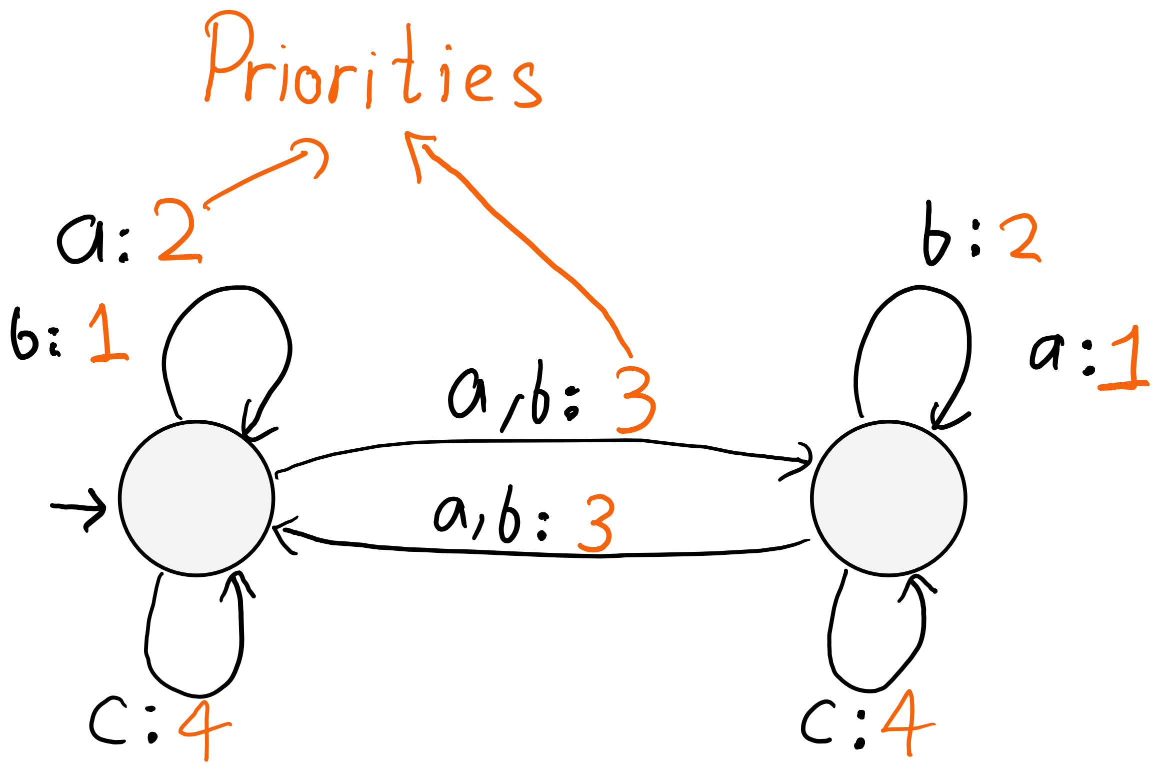


Accepting run: Sequence of priorities satisfies the parity condition.

Accepting word: If the automaton has an accepting run on it.

Parity Automata

Input: $w \in \{a,b,c\}^{\mathbb{N}}$



Accepting run:
Sequence of priorities satisfies the parity condition.

Accepting word: If the automaton has an accepting run on it.

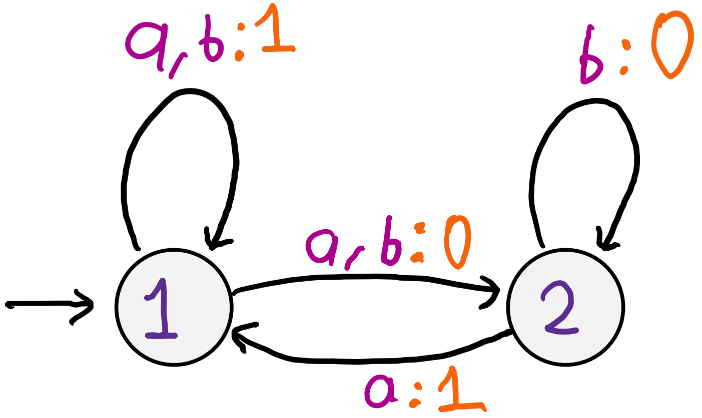
Language: Set of accepting words.

History-Deterministic Automata

Nondeterminism that arises while reading a word can be resolved based only on the prefix read so far.

Henzinger, Piterman 2006

History-Determinism Game



Starts at $\to 1$

Adam selects letter a_i

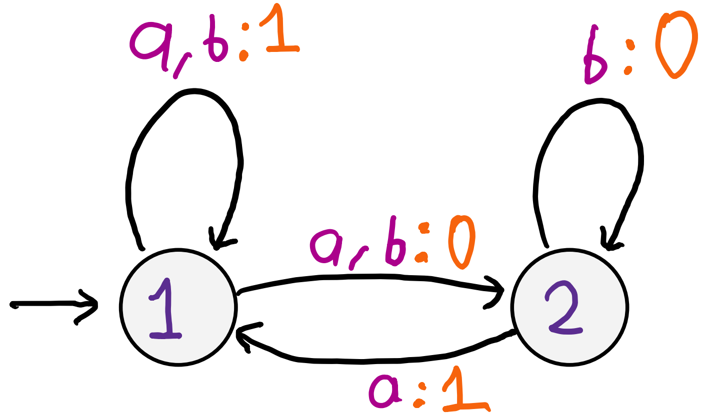
Eve selects transition $q_i \xrightarrow{a_i} q_{i+1}$

H.D. Game

Adam

Eve 1

History-Determinism Game



Starts at → 1

Adam selects letter a_i

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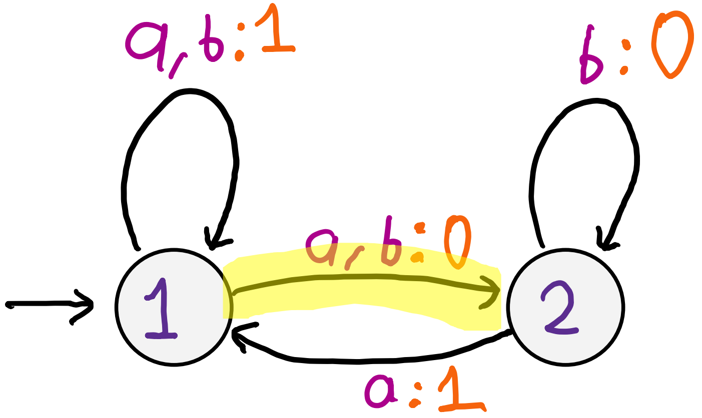
H.D. Game

Adam	a
Eve	1

History-Determinism Game



Starts at $\rightarrow 1$

Adam selects letter a_i

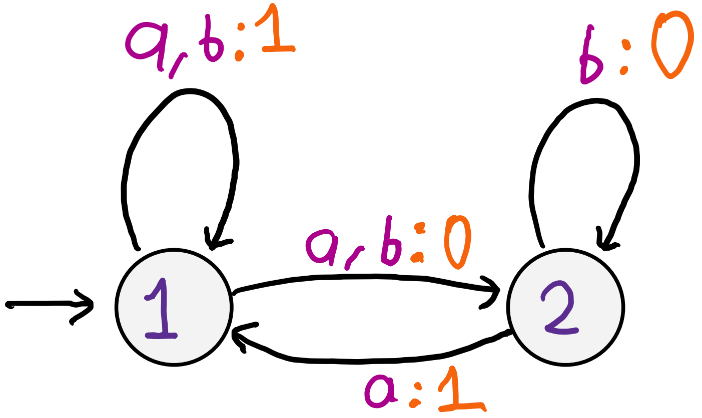
Eve selects transition $q_i \xrightarrow{a_i} q_{i+1}$

H.D. Game

Adam a

Eve $1 \xrightarrow{a} 2$

History-Determinism Game



Starts at $\rightarrow 1$

Adam selects letter a_i

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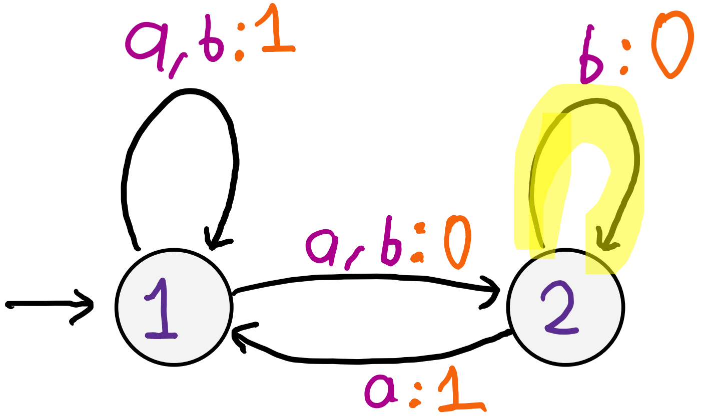
H.D. Game

Adam a b

Eve $1 \xrightarrow{a} 2$

History-Determinism Game



Starts at $\rightarrow (1)$

Adam selects letter a_i

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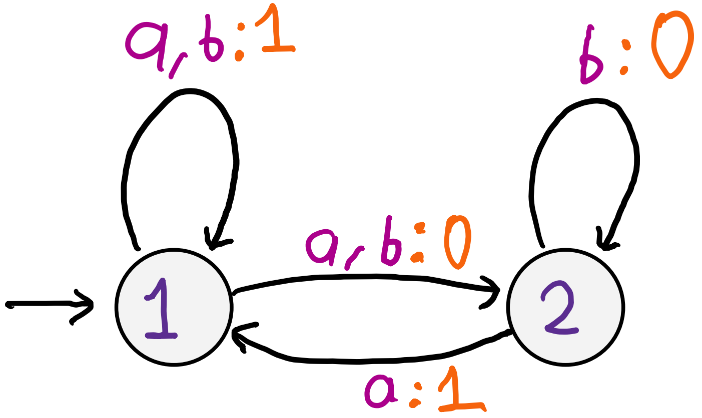
H.D. Game

Adam	a	b
Eve	$1 \xrightarrow{a} 2$	$\xrightarrow{b} 2$

History-Determinism Game



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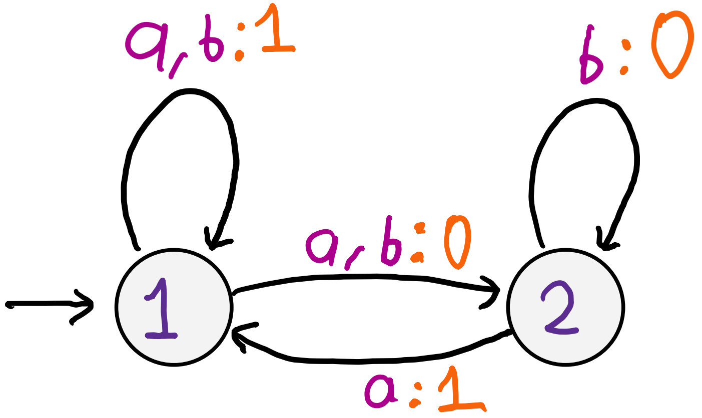
H.D. Game

Adam	a	b	a	...
Eve	$1 \xrightarrow{a} 2$	$\xrightarrow{b} 2$	$\xrightarrow{a} 1$	$\rightarrow \dots$

History-Determinism Game



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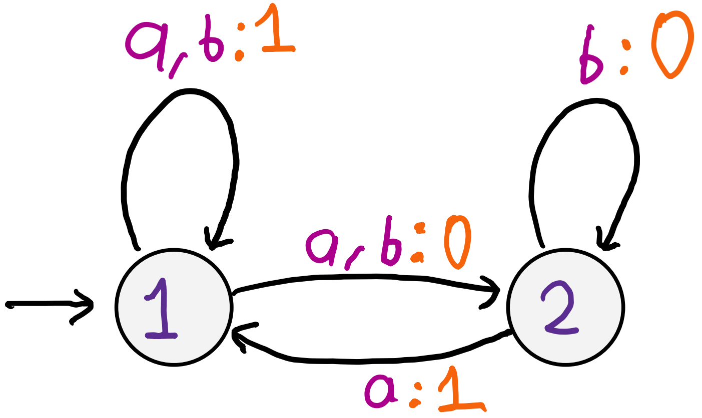
Winning cond$^{n.}$ for Eve:

Construct an accepting run if Adam's word is accepting.

H.D. Game

Adam	a	b	a	...
Eve	$1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \dots$			

History-Determinism Game



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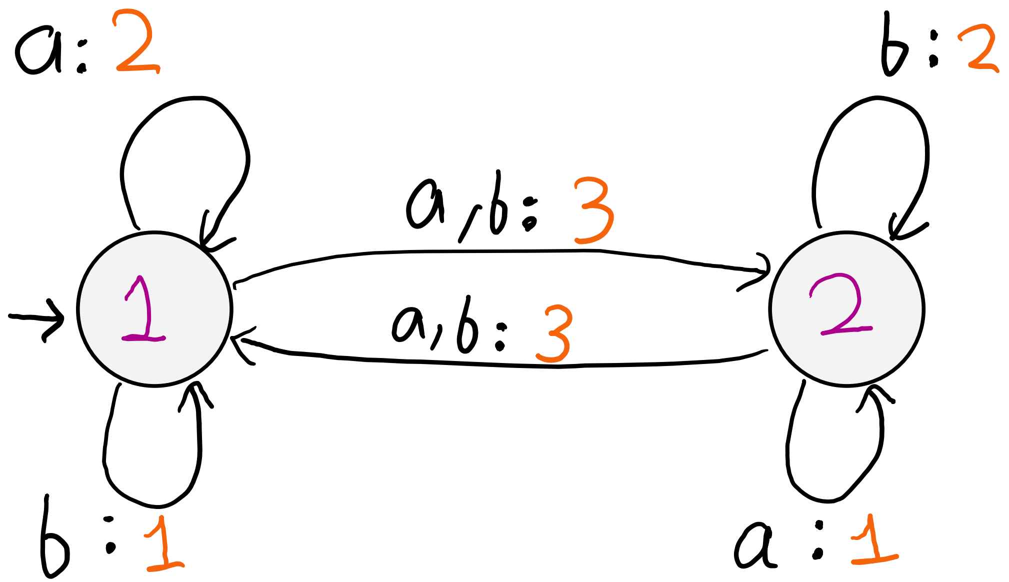
Adam	a	b	a	...
Eve	$1 \xrightarrow{a} 2$	$\xrightarrow{b} 2$	$\xrightarrow{a} 1$	$\rightarrow \ldots$

Winning cond$^{n.}$ for Eve:

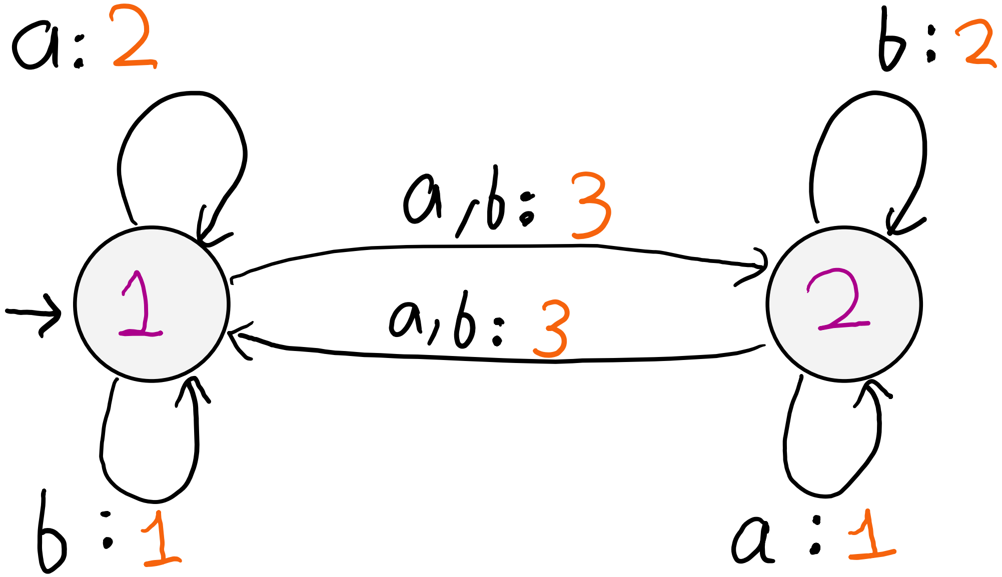
Construct an accepting run if Adam's word is accepting.

HD Automata: Eve has a winning strategy

Non-example

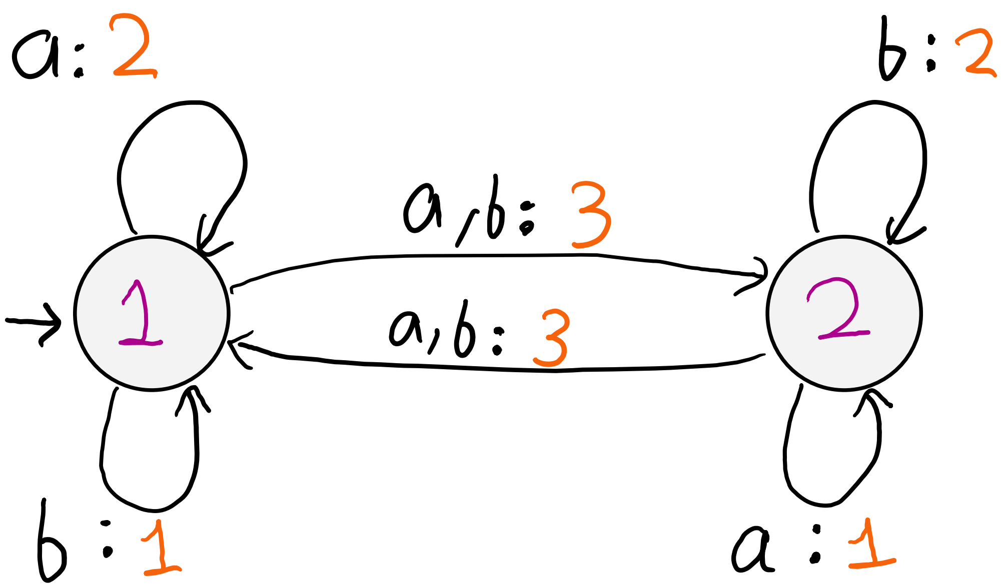


Non-example



$$L = (a+b)^{\omega}$$

Non-example



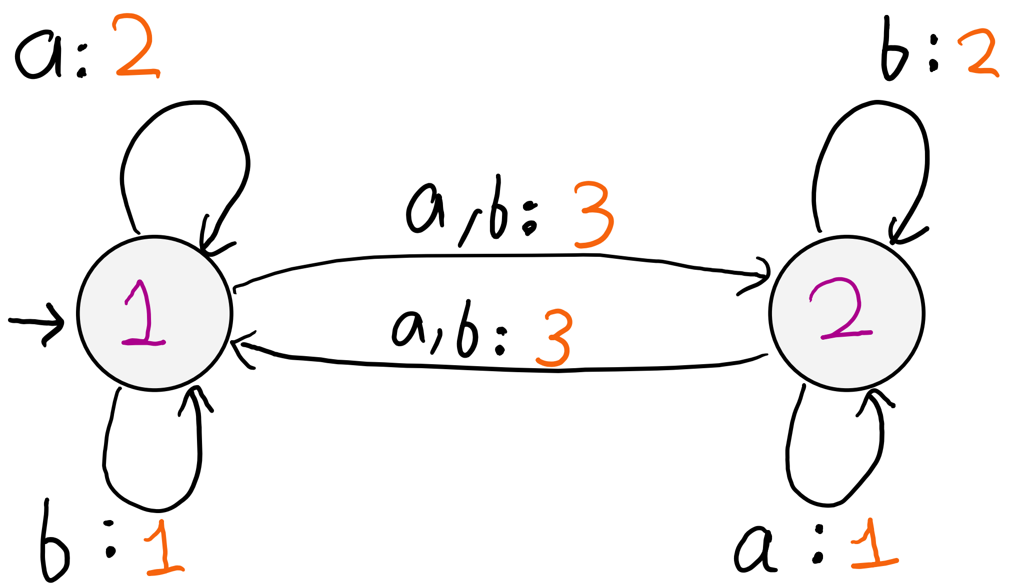
$L = (a+b)^\omega$

HD Game

Adam:

Eve: 1

Non-example



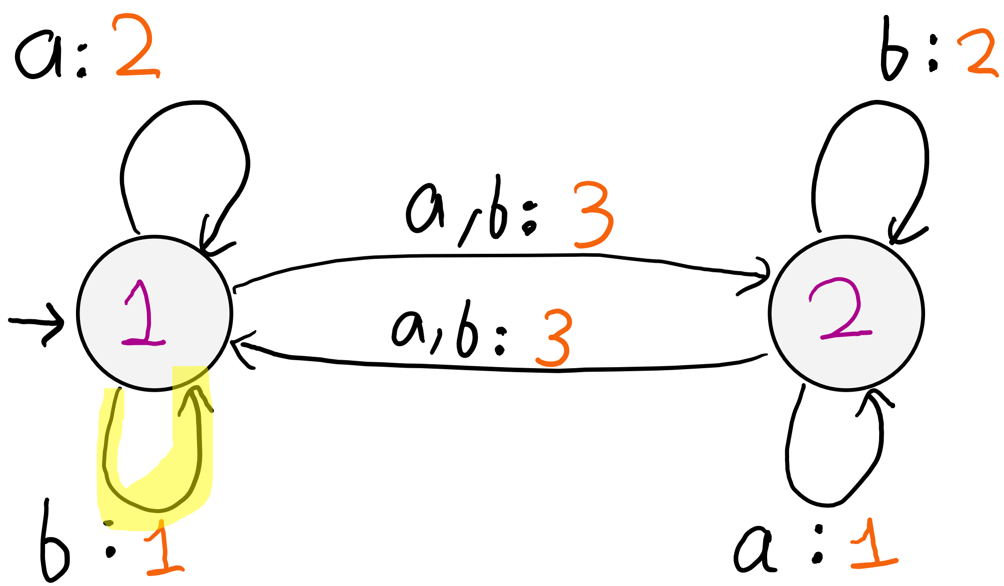
$L = (a+b)^\omega$

HD Game

Adam: b

Eve: 1

Non-example



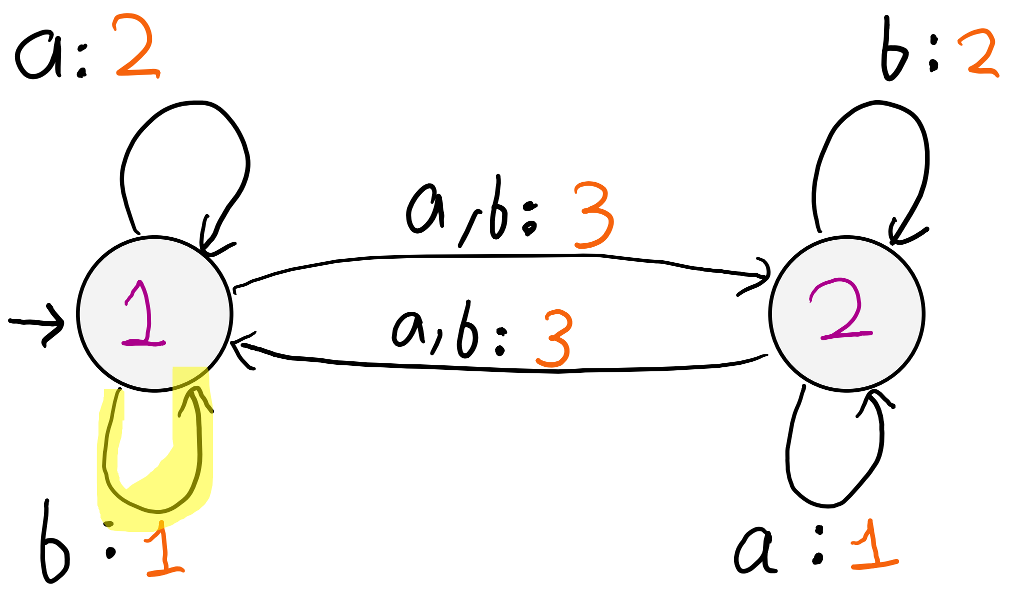
$L = (a+b)^\omega$

HD Game

Adam: b

Eve: 1 ⟶ 1

Non-example



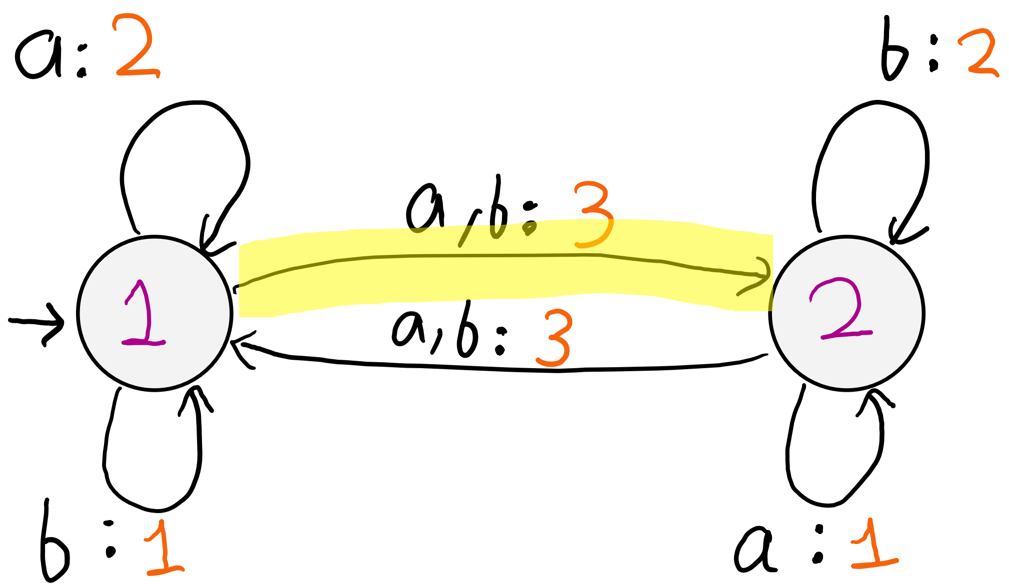
$L = (a+b)^{\omega}$

HD Game

Adam: b b ... b

Eve: $1 \longrightarrow 1 \rightarrow \ldots \rightarrow 1$

Non-example



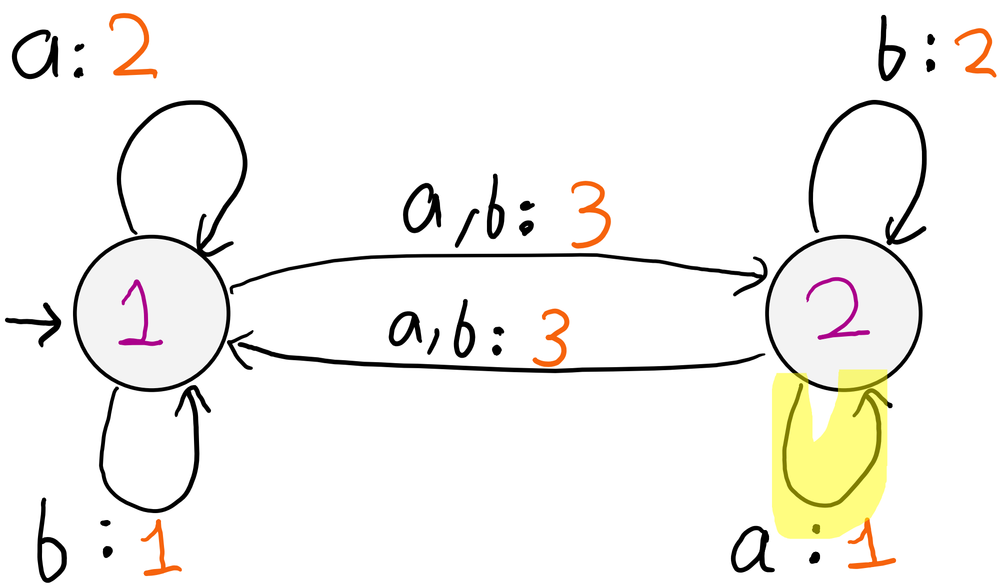
$$L = (a+b)^{\omega}$$

HD Game

Adam: b b ... b

Eve: 1 ⟶ 1 → ... → 1 → 2

Non-example



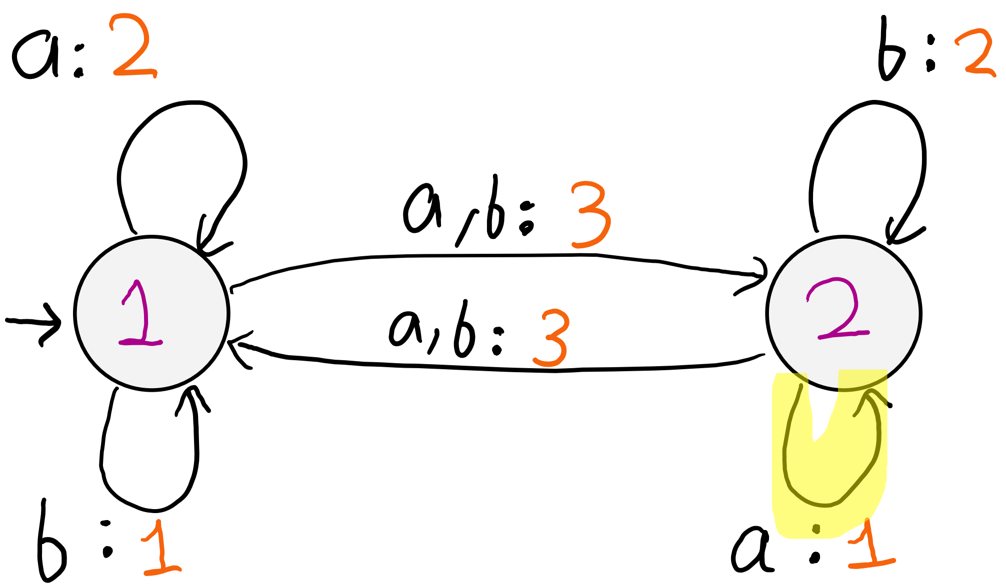
$L = (a+b)^\omega$

HD Game

Adam: $b \quad b \ldots b \quad a \ldots a$

Eve: $1 \longrightarrow 1 \rightarrow \ldots \rightarrow 1 \rightarrow 2 \rightarrow \ldots \rightarrow 2$

Non-example



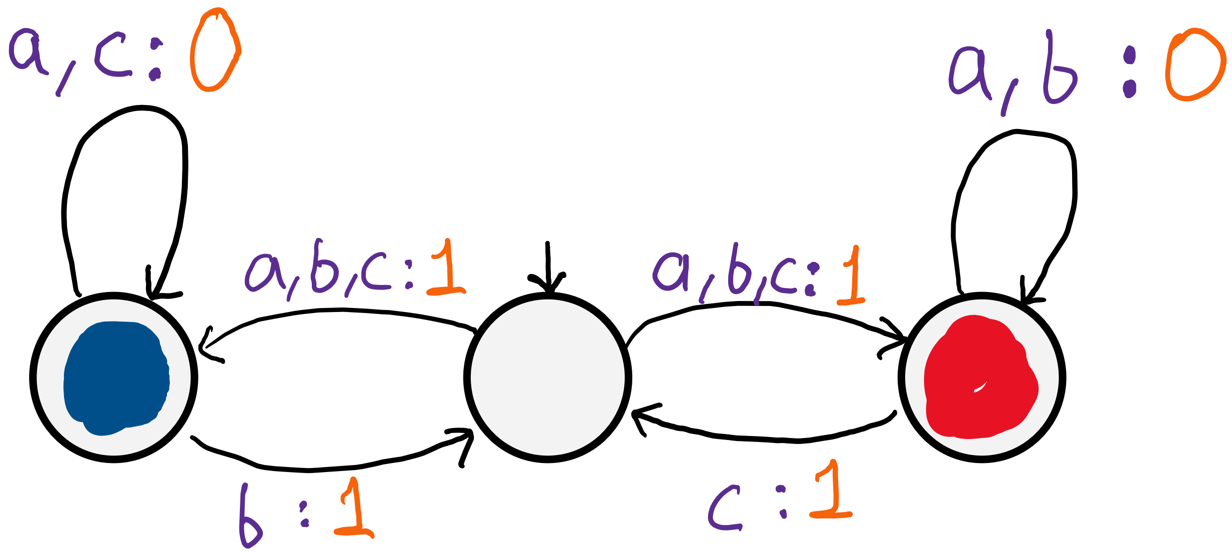
$$L = (a+b)^\omega$$

HD Game

Adam: $b \quad b \dots b \quad a \dots a \quad \in L$

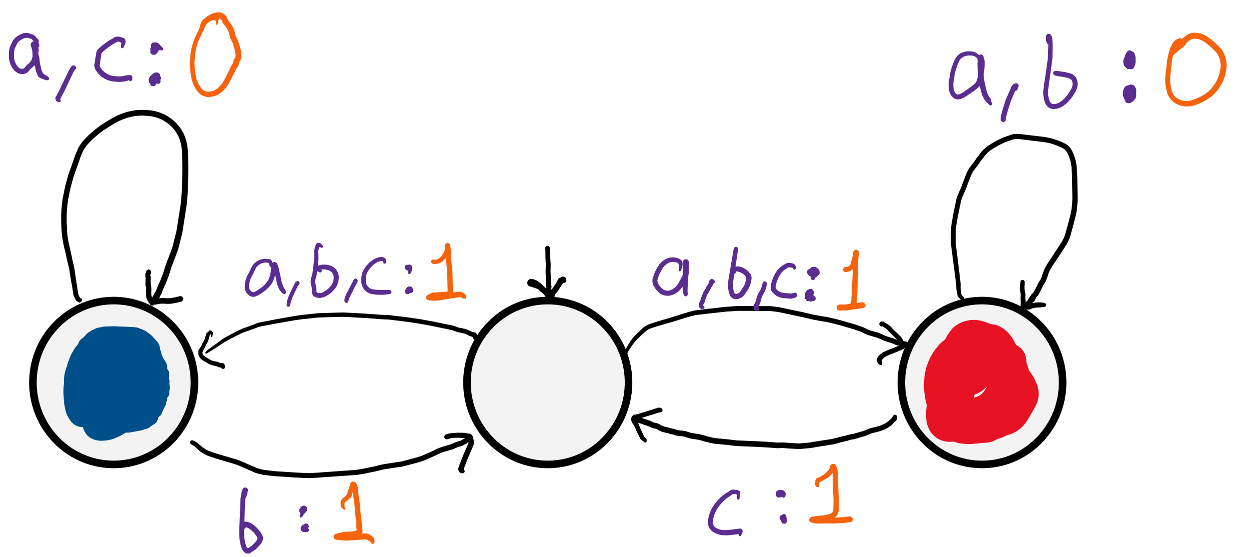
Eve: $1 \longrightarrow 1 \to \dots \to 1 \to 2 \to \dots \to 2$ ✗

Example: coBüchi automata



Accepting Condition: Finitely many 1's

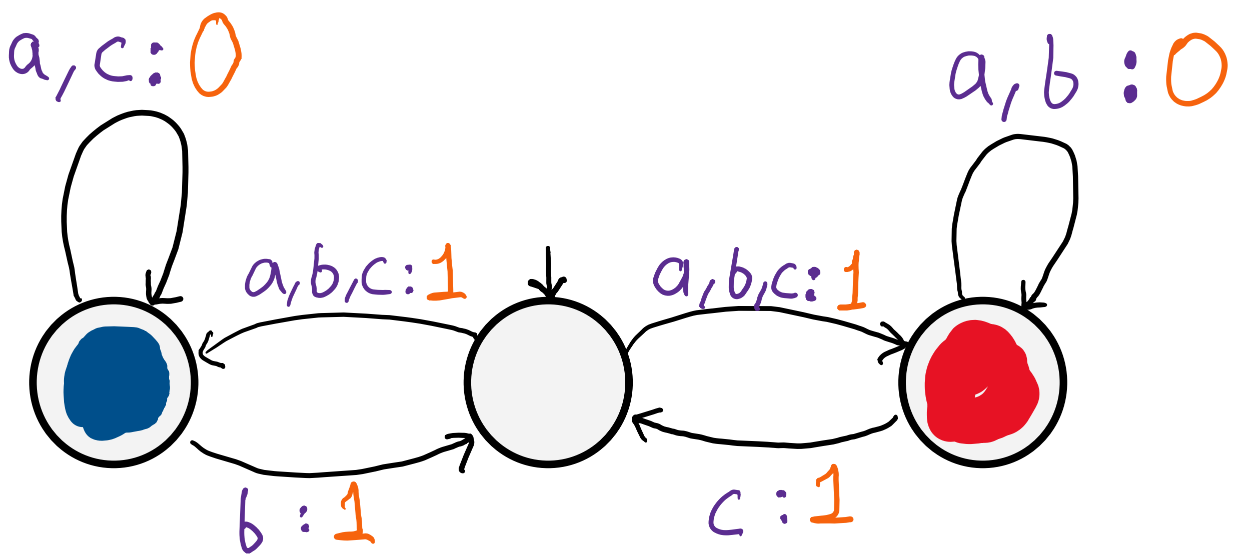
Example: coBüchi automata



Accepting Condition: Finitely many 1's

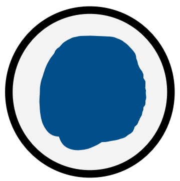
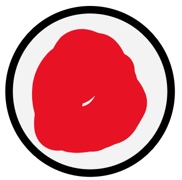
$L = \{ w \mid w$ contains finitely many b's or finitely many c's $\}$

Example: coBüchi automata



HD game strategy:

Alternate between

 and 

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Why History-Determinism?

Language Inclusion

I S

Implementation Specification

$$L(I) \subseteq L(S)?$$

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Implementation Specification

$$L(I) \subseteq L(S)?$$

Parity Automata: PSPACE

Why History-Determinism?

Language Inclusion

I $\qquad$ $S \xrightarrow{\text{(Folklore)}}$ If S is HD:

Implementation $\qquad$ Specification

$L(I) \subseteq L(S)$? $\qquad$ Equivalent to asking:

Does S simulate I ?

Parity Automata: PSPACE $\rightsquigarrow$ NP

(Fair) Simulation Game

Automata I , S

Starts at $\rightarrow p_0$, $\rightarrow s_0$

In round i :

1. Adam selects a_i
2. Adam selects $p_i \xrightarrow{a_i} p_{i+1}$ in I
3. Eve selects $s_i \xrightarrow{a_i} s_{i+1}$ in S

Henzinger, Kupferman, Rajamani '97

(Fair) Simulation Game

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$$w = \quad a_0 \quad a_1 \quad a_2 \quad a_3 \dots$$

$$\rho_A = \to p_0 \to p_1 \to p_2 \to p_3 \to \dots$$

$$\rho_E = \to s_0 \to s_1 \to s_2 \to s_3 \to \dots$$

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$$w = a_0 \quad a_1 \quad a_2 \quad a_3 \dots$$

$$\rho_A = \to p_0 \to p_1 \to p_2 \to p_3 \to \dots$$

$$\rho_E = \to s_0 \to s_1 \to s_2 \to s_3 \to \dots$$

Winning condition for Eve:

ρ_A is accepting $\Rightarrow$ ρ_E is accepting

Henzinger, Kupferman, Rajamani '97

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$$\rho_A = \to p_0 \to p_1 \to p_2 \to p_3 \to \cdots$$

$$\rho_E = \to s_0 \to s_1 \to s_2 \to s_3 \to \cdots$$

Winning condition for Eve:

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S simulates I if Eve has a winning strategy.

$I \hookrightarrow S$

Henzinger, Kupferman, Rajamani '97

Why History-Determinism?

* Model-checking: Inclusion reduces to simulation

Lemma : If S is HD, then for all I,
$L(I) \subseteq L(S) \Leftrightarrow S$ simulates I

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Proof : $\Leftarrow$: Clear.

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Proof: $\Leftarrow$: Clear.

$\Rightarrow$: Winning strategy for Eve in simulation game:

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Lemma: If S is HD, then for all I,

$$L(I) \subseteq L(S) \Leftrightarrow S \text{ simulates } I$$

Proof: $\Leftarrow$: Clear.

$\Rightarrow$: Winning strategy for Eve in simulation game: select transitions using Eve's winning strategy in HD game on S.

Why History-Determinism?

* Model-checking: Inclusion reduces to simulation

Lemma : If S is HD, then for all I,
$L(I) \subseteq L(S) \Leftrightarrow S$ simulates I

Corollary: Deciding $L(I) \subseteq L(S)$ is in NP
if S is HD.

Why History-Determinism?

* Model-checking: Inclusion reduces to simulation

Lemma : If S is HD, then for all I,
$L(I) \subseteq L(S) \Leftrightarrow S$ simulates I

Theorem 1: Deciding $L(I) \subseteq L(S)$ is in ~~NP~~ quasi-polynomial time
if S is HD.

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* Model-checking: Inclusion reduces to simulation
* Good-for-games

[Henzinger, Piterman'06]

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* Exponentially more succinct than deterministic parity automata

  [Kuperberg, Skrzypczak'15]

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But : No known tractable algorithm to construct HD automata

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But : No known tractable algorithm to construct HD automata

Nevertheless: Still useful in synthesis [Khalimov, Ehlers'24]

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Given a parity automaton S, is S HD?

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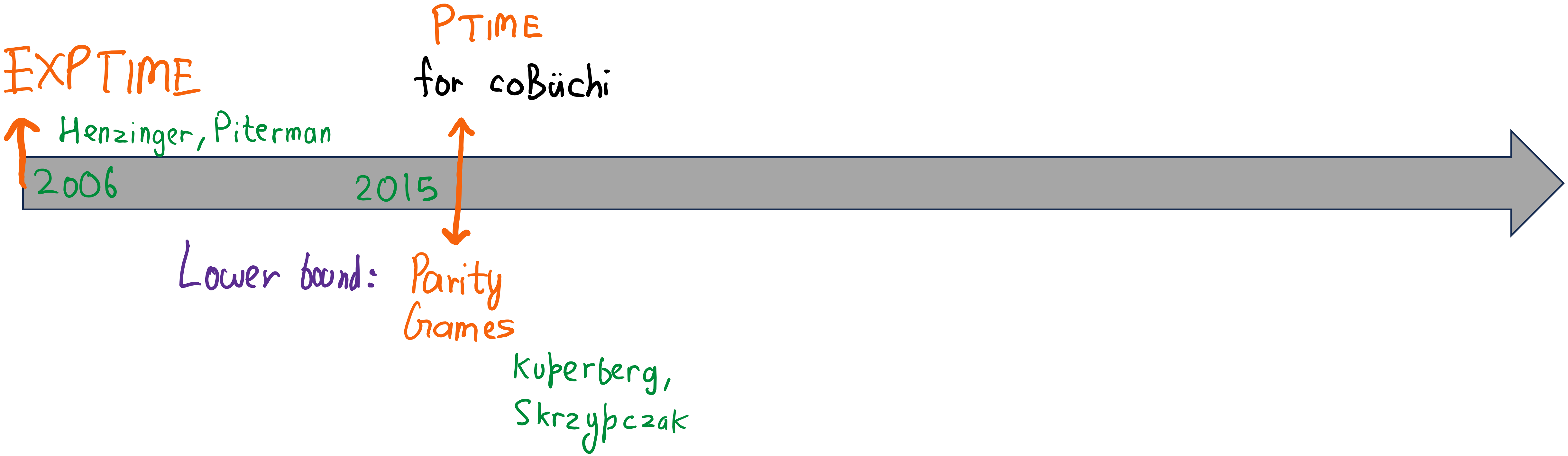

Recognising HD Parity Automata

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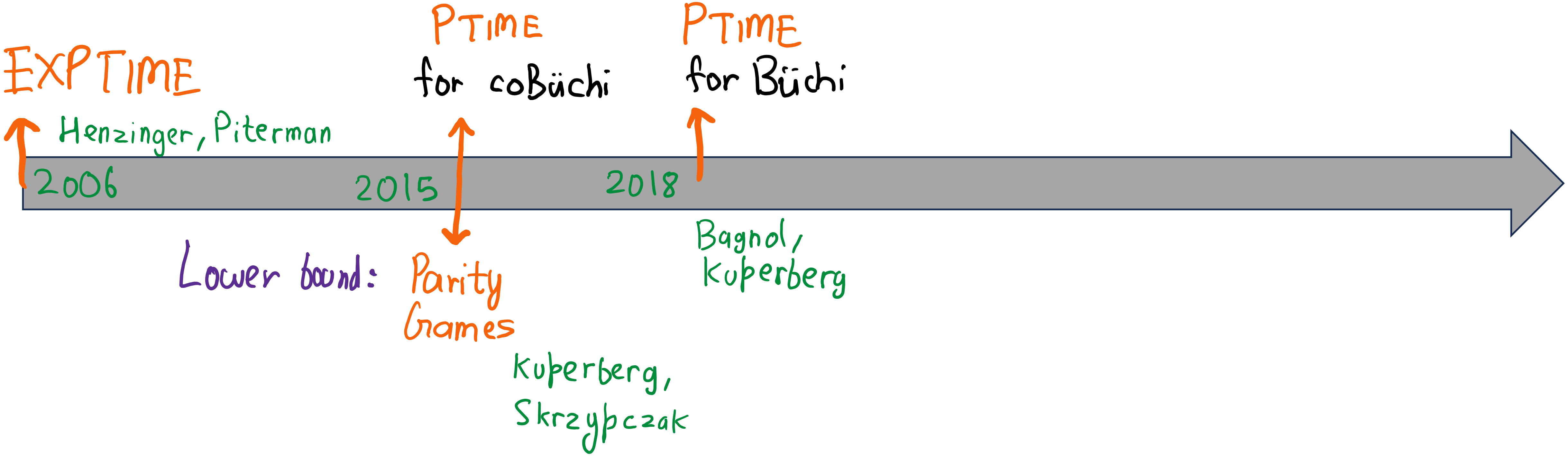

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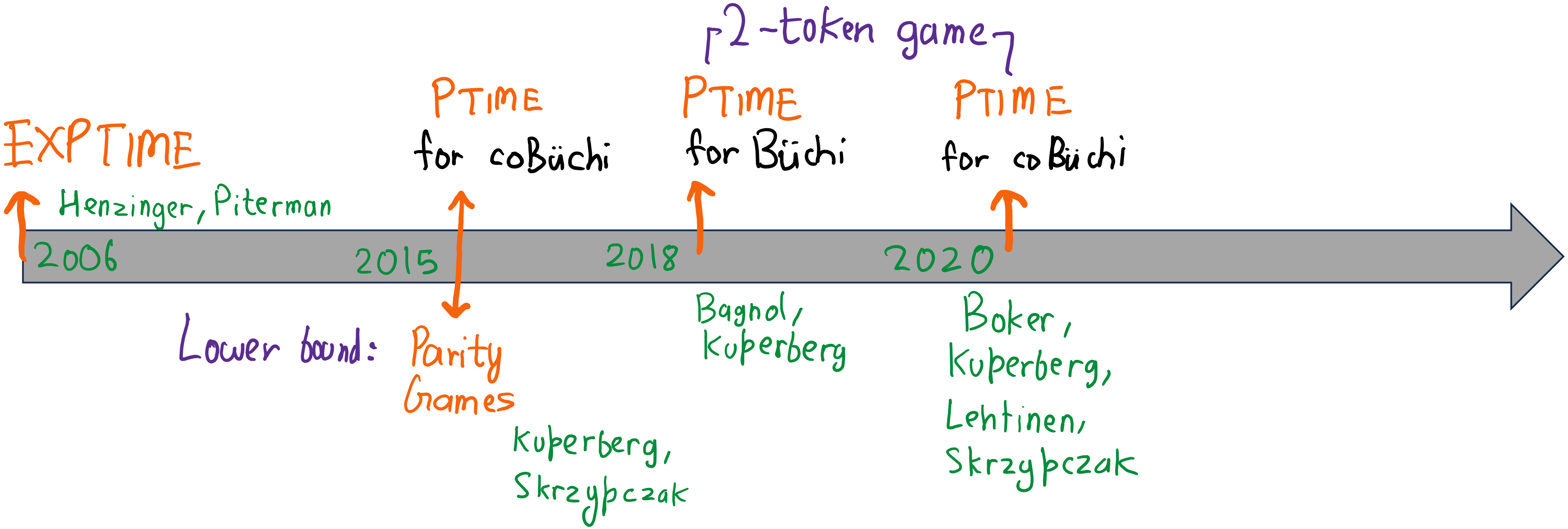


Recognising HD Parity Automata

Given a parity automaton $\mathcal{S}$, is $\mathcal{S}$ HD?

EXPTIME

Henzinger, Piterman

2006

PTIME for coBüchi

2015

Lower bound: Parity Games

Kuperberg, Skrzypczak

2-token game

PTIME for Büchi

2018

Bagnol, Kuperberg

PTIME for coBüchi

2020

Boker, Kuperberg, Lehtinen, Skrzypczak

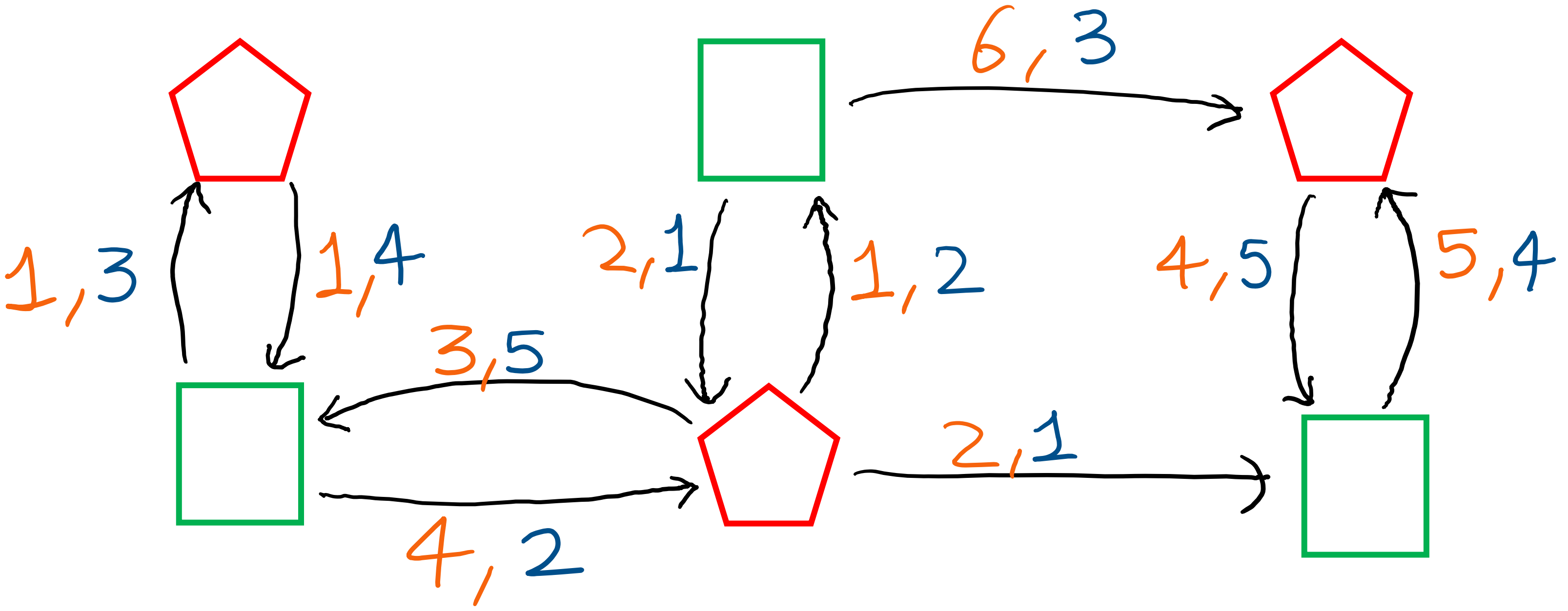
2024

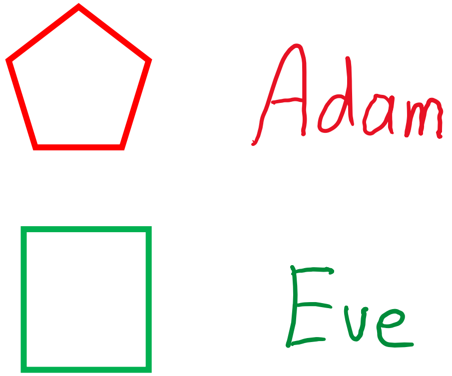
Thm.: NP-hard.

Theorem: Checking history-determinism is NP-hard for parity automata.

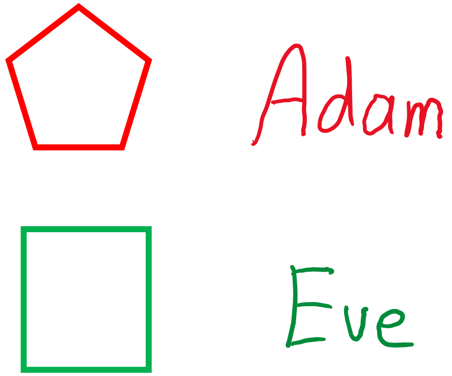
Proof: Reduction from 2-D parity game

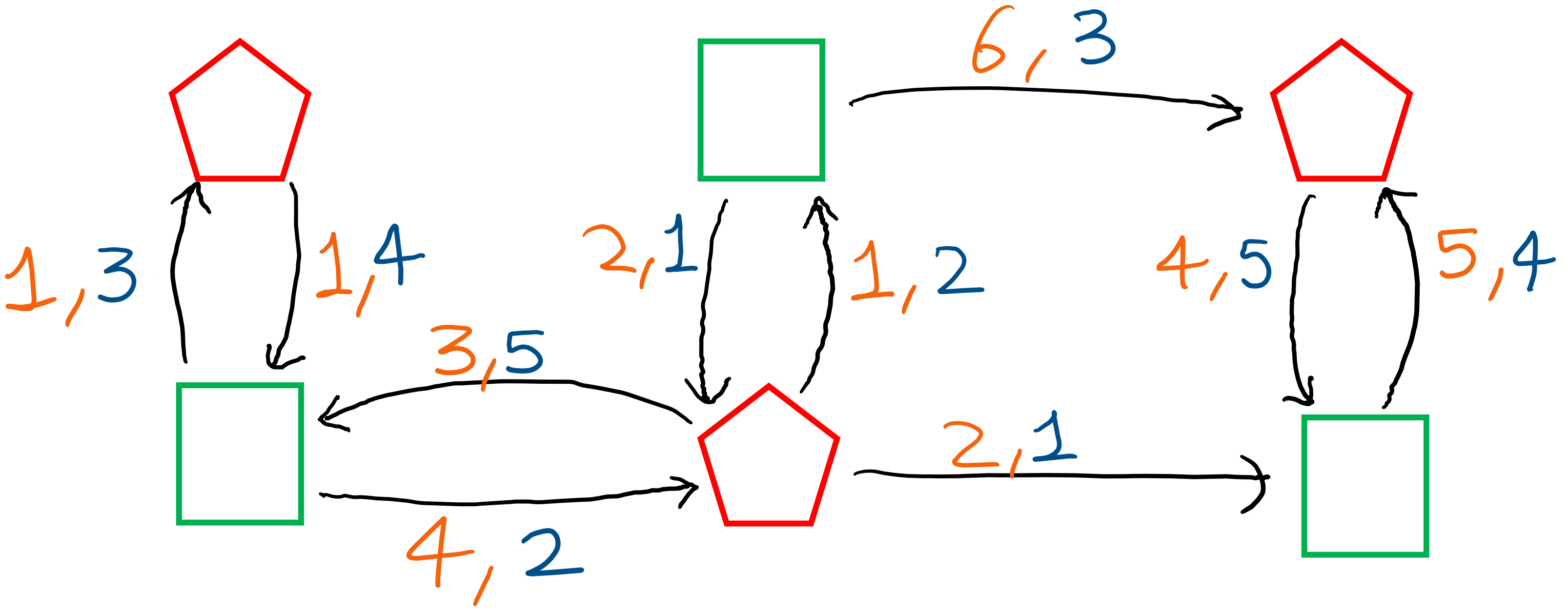
2-D Parity Games





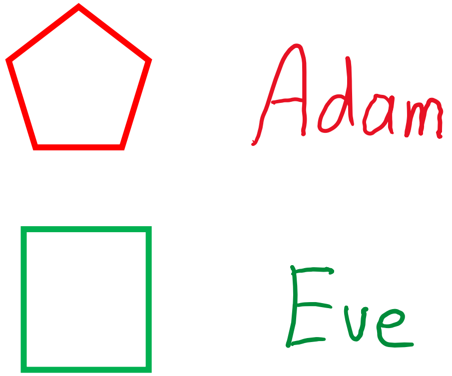
2-D Parity Games

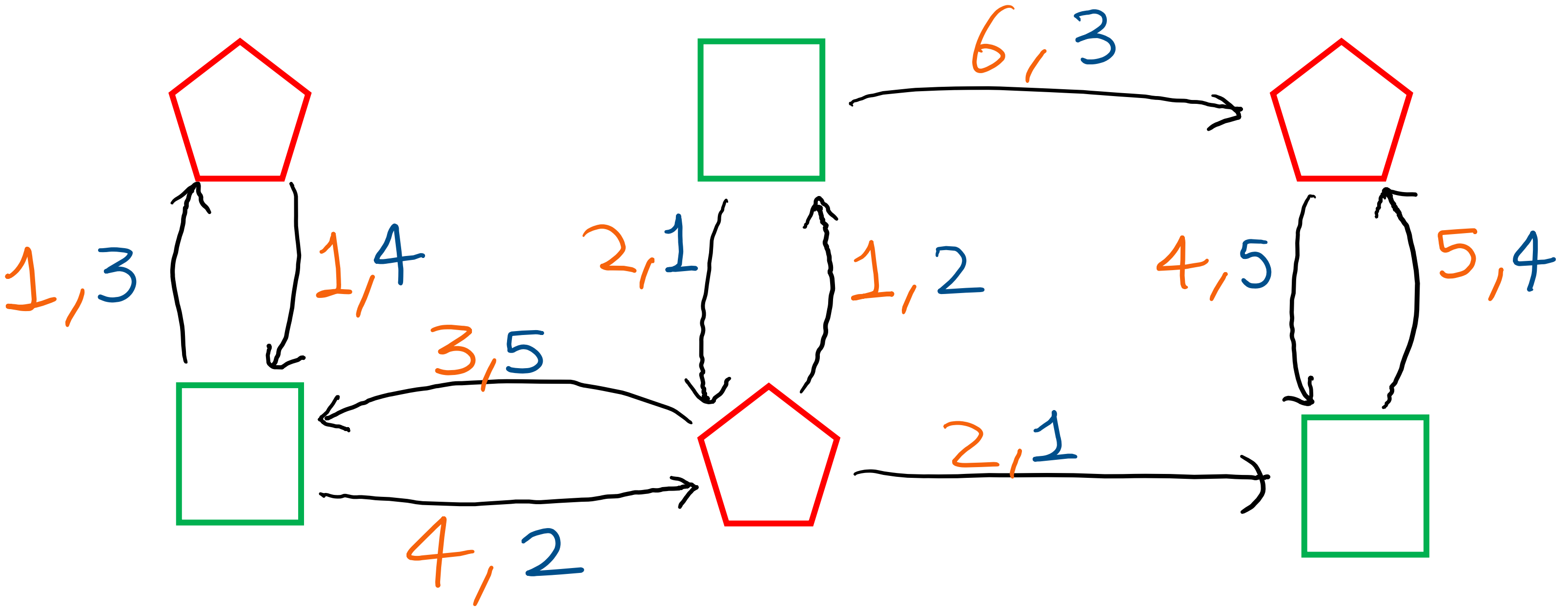




(1,3) (1,4) (4,2) (2,1) (6,3) ...

2-D Parity Games





(1,3) (1,4) (4,2) (2,1) (6,3) ...

Winning cond^{n} for Eve: Play satisfies Orange or Blue parity cond^{n}.

Chatterjee, Henzinger, Piterman '05

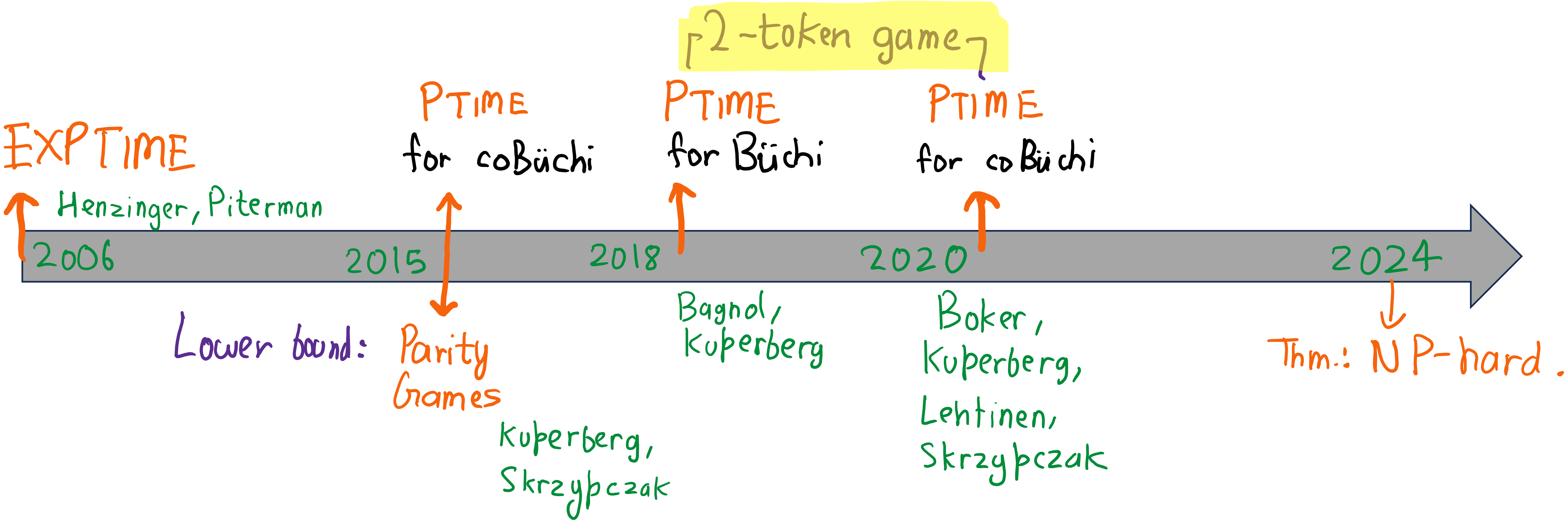
Deciding if Eve wins a 2-D parity game is NP-complete.

Theorem: Checking history-determinism is NP-hard for parity automata.

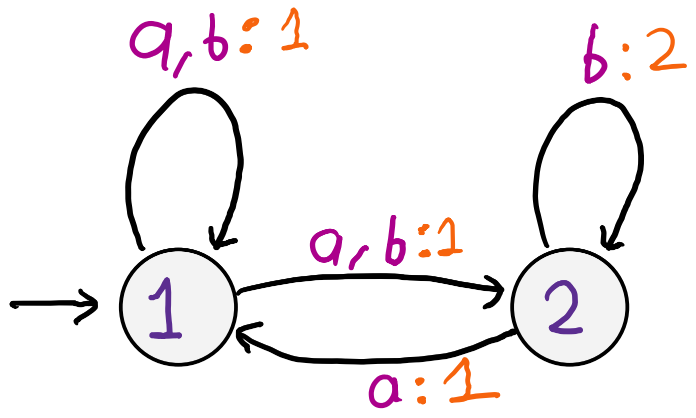
II. Token Games

Recognising HD Parity Automata

Given a parity automaton S, is S HD?



2-Token Games



Starts at $\to 1, \to 1, \to 1$

Adam selects letter a_i

Eve selects transition $q_i \xrightarrow{a_i} q_{i+1}$

Adam selects transitions $p^1_i \xrightarrow{a_i} p^1_{i+1}$

$p^2_i \xrightarrow{a_i} p^2_{i+1}$

2-Token Game

Adam	
Eve	1
Adam	1
Adam	1

2-Token Games

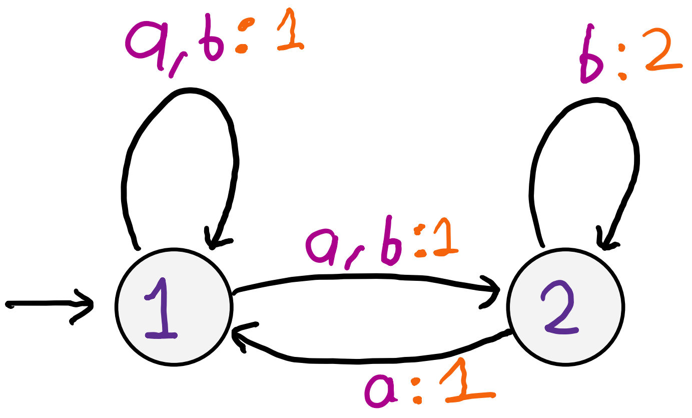
Starts at $\to 1$, $\to 1$, $\to 1$

Adam selects letter a_i

Eve selects transition $q_i \xrightarrow{a_i} q_{i+1}$

Adam selects transitions $p_i^1 \xrightarrow{a_i} p_{i+1}^1$

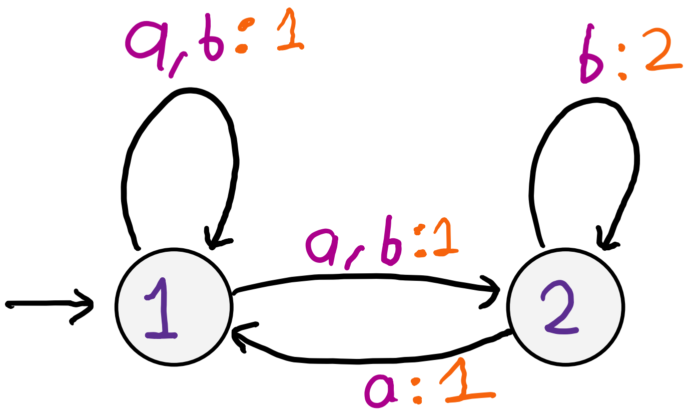
$p_i^2 \xrightarrow{a_i} p_{i+1}^2$



2-Token Game

Adam	a	b	a	...
Eve	$1 \xrightarrow{a} 2$	$\xrightarrow{b} 2$	$\xrightarrow{a} 1$	$\to \dots$
Adam	$1 \xrightarrow{a} 2$	$\xrightarrow{b} 2$	$\xrightarrow{a} 1$	$\to \dots$
Adam	$1 \xrightarrow{a} 1$	$\xrightarrow{b} 1$	$\xrightarrow{a} 1$	$\to \dots$

2-Token Games



Starts at $\to 1, \to 1, \to 1$

Adam selects letter a_i

Eve selects transition $q_i \xrightarrow{a_i} q_{i+1}$

Adam selects transitions $p_i^1 \xrightarrow{a_i} p_{i+1}^1$

$p_i^2 \xrightarrow{a_i} p_{i+1}^2$

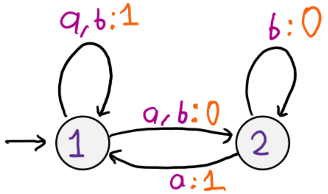
Winning condⁿ for Eve: Construct an accepting run if one of Adam's run is accepting.

2-Token Game

Adam	a	b	a	...
Eve	$1 \xrightarrow{a} 2$	$\xrightarrow{b} 2$	$\xrightarrow{a} 1$	$\to \dots$
Adam	$1 \xrightarrow{a} 2$	$\xrightarrow{b} 2$	$\xrightarrow{a} 1$	$\to \dots$
Adam	$1 \xrightarrow{a} 1$	$\xrightarrow{b} 1$	$\xrightarrow{a} 1$	$\to \dots$

HD Game v/s 2- Token Game

History- Determinism Game



Starts at $\rightarrow 1$

Adam selects letter a_i

Eve selects transition $q_i \xrightarrow{a_i} q_{i+1}$

Winning cond^n for Eve:

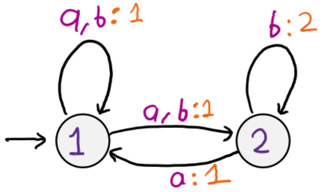
Construct an accepting run if Adam's word is accepting.

HD Automata: Eve has a winning strategy

H.D. Game

Adam	a	b	a	...
Eve	$1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \dots$			

2- Token Games



Starts at $\rightarrow 1, \rightarrow 1, \rightarrow 1$

Adam selects letter a_i

Eve selects transition $q_i \xrightarrow{a_i} q_{i+1}$

Adam selects transitions $p_i^1 \xrightarrow{a_i} p_{i+1}^1$

$p_i^2 \xrightarrow{a_i} p_{i+1}^2$

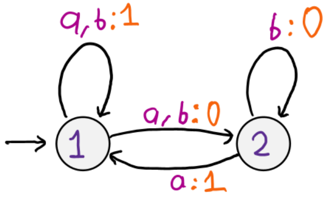
Winning cond^n for Eve: Construct an accepting run if one of Adam's run is accepting.

2- Token Game

Adam	a	b	a	...
Eve	$1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \dots$			
Adam	$1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \dots$			
Adam	$1 \xrightarrow{a} 1 \xrightarrow{b} 1 \xrightarrow{a} 1 \rightarrow \dots$			

HD Game v/s 2- Token Game

History- Determinism Game



Starts at $\rightarrow (1)$

Adam selects letter a_i

Eve selects transition $q_i \xrightarrow{a_i} q_{i+1}$

Winning cond^n. for Eve:

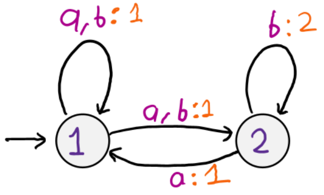
Construct an accepting run if Adam's word is accepting.

HD Automata: Eve has a winning strategy

H.D. Game

Adam	a	b	a	...
Eve	$1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \dots$			

2- Token Games



Starts at $\rightarrow (1), \rightarrow (1), \rightarrow (1)$

Adam selects letter a_i

Eve selects transition $q_i \xrightarrow{a_i} q_{i+1}$

Adam selects transitions $p_i^1 \xrightarrow{a_i} p_{i+1}^1$

$p_i^2 \xrightarrow{a_i} p_{i+1}^2$

Winning cond^n. for Eve: Construct an accepting run if one of Adam's run is accepting.

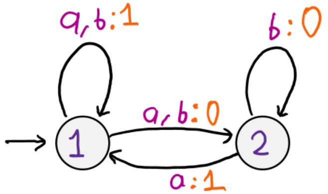
2- Token Game

Adam	a	b	a	...
Eve	$1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \dots$			
Adam	$1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \dots$			
Adam	$1 \xrightarrow{a} 1 \xrightarrow{b} 1 \xrightarrow{a} 1 \rightarrow \dots$			

Eve wins HD Game $\Rightarrow$ Eve wins 2-token game

HD Game v/s 2-Token Game

History-Determinism Game



Starts at $\rightarrow 1$

Adam selects letter a_i

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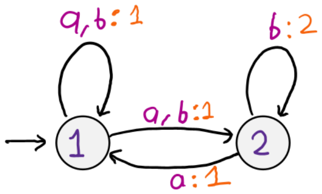
Construct an accepting run if Adam's word is accepting.

HD Automata: Eve has a winning strategy

H.D. Game

Adam	a	b	a	...
Eve	$1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \ldots$			

2-Token Games



Starts at $\rightarrow 1, \rightarrow 1, \rightarrow 1$

Adam selects letter a_i

Eve selects transition $q_i \xrightarrow{a_i} q_{i+1}$

Adam selects transitions $p_i^1 \xrightarrow{a_i} p_{i+1}^1$, $p_i^2 \xrightarrow{a_i} p_{i+1}^2$

Winning cond^n for Eve: Construct an accepting run if one of Adam's run is accepting.

2-Token Game

Adam	a	b	a	...
Eve	$1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \ldots$			
Adam	$1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \ldots$			
Adam	$1 \xrightarrow{a} 1 \xrightarrow{b} 1 \xrightarrow{a} 1 \rightarrow \ldots$			

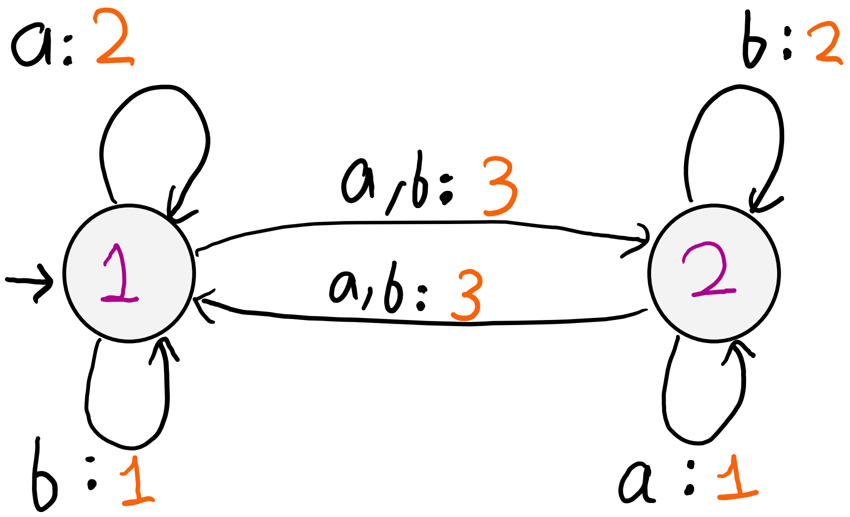
Eve wins HD Game $\Rightarrow$ Eve wins 2-token game

$\Leftarrow$

for Büchi, co Büchi

Why 2-tokens?

1. 1-token game is not enough:



Why 2-tokens?

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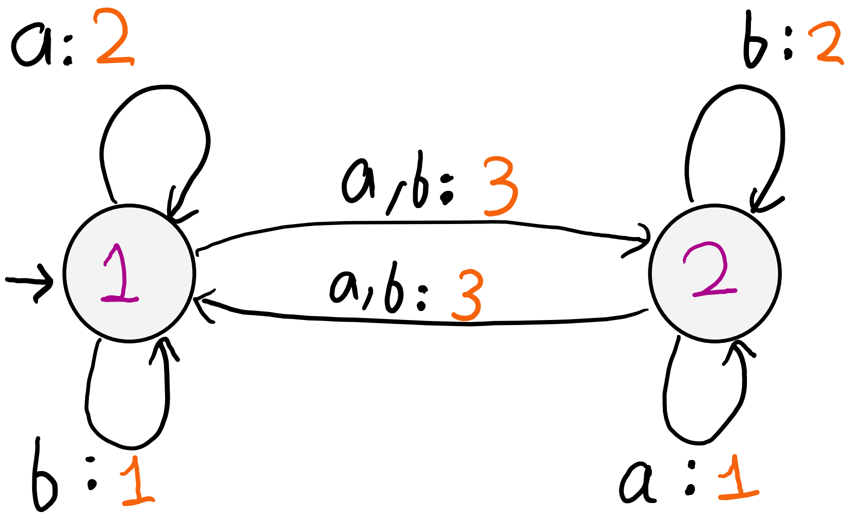
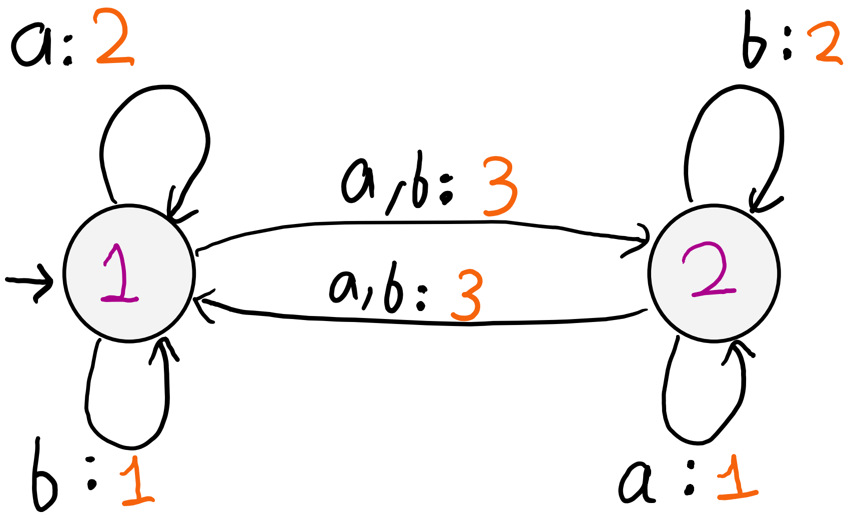


Not HD.

Eve wins 1-token game.

Why 2-tokens?

1. 1-token game is not enough:



Not HD.

Eve wins 1-token game.

Strategy: Go to Adam's state.

Why 2-tokens?

1. 1-token game is not enough:

2. 2-tokens are plenty!

Lemma Bagnol, Kuperberg '18

Eve wins 2-token game $\iff$ Eve wins k-token game for all $k \geqslant 1$.

Token Games for checking History-Determinism

Lemma Bagnol, Kuperberg '18

Eve wins 2-token game $\iff$ Eve wins k-token game for all $k \geq 1$.

Token Games for checking History-Determinism

Lemma Bagnol, Kuperberg '18

Eve wins 2-token game $\iff$ Eve wins k-token game for all $k \geq 1$.

Theorem Bagnol, Kuperberg '18

For any Büchi automaton A, Eve wins 2-token game on A iff A is HD

Token Games for checking History-Determinism

Theorem Boker, Kuperberg, Lehtinen, Skrzypczak'20

For any coBüchi automaton $\mathcal{A}$, Eve wins 2-token game on $\mathcal{A}$ iff $\mathcal{A}$ is HD.

Token Games for checking History-Determinism

Theorem Boker, Kuperberg, Lehtinen, Skrzypczak '20

For any coBüchi automaton A, Eve wins 2-token game on A iff A is HD.

2-token conjecture: For any parity automaton A, Eve wins 2-token game on A iff A is HD.

Token Games for checking History-Determinism

Theorem Boker, Kuperberg, Lehtinen, Skrzypczak '20

For any coBüchi automaton A, Eve wins 2-token game on A iff A is HD.

2-token conjecture: For any parity automaton A, Eve wins 2-token game on A iff A is HD.

→ **Implies:** HD parity automata can be recognised in PSPACE, and in PTIME if parity index is fixed.

3. Lookahead Games

1-Token Game

Automaton S

$w = a_0 \quad a_1 \quad a_2 \quad a_3 \dots$

$\rho_E = \to s_0 \to s_1 \to s_2 \to s_3 \to \dots$

$\rho_A = \to s_0 \to s'_1 \to s'_2 \to s'_3 \to \dots$

Starts at $\to s_0$, $\to s_0$

In round i:

1. Adam selects a_i
2. Eve selects $s_i \xrightarrow{a_i} s_{i+1}$
3. Adam selects $s'_i \xrightarrow{a_i} s'_{i+1}$

Winning condition for Eve: ρ_A is accepting $\Rightarrow$ ρ_E is accepting

1-token game v/s Simulation game

1-Token Game

$w = a_0 \quad a_1 \quad a_2 \quad a_3 \ldots$

$\rho_E = \to s_0 \to s_1 \to s_2 \to s_3 \to \ldots$

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Automaton S

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Winning condition for Eve : ρ_A is accepting $\Rightarrow \rho_E$ is accepting

(Fair) Simulation Game

$w = a_0 \quad a_1 \quad a_2 \quad a_3 \ldots$

$\rho_A = \to p_0 \to p_1 \to p_2 \to p_3 \to \ldots$

$\rho_E = \to s_0 \to s_1 \to s_2 \to s_3 \to \ldots$

Automata I , S

Starts at $\to p_0$, $\to s_0$

In round i :

1. Adam selects a_i
2. Adam selects $p_i \xrightarrow{a_i} p_{i+1}$
3. Eve selects $s_i \xrightarrow{a_i} s_{i+1}$

Winning condition for Eve : ρ_A is accepting $\Rightarrow \rho_E$ is accepting

1-token game v/s Simulation game

1-Token Game

$w = a_0 \quad a_1 \quad a_2 \quad a_3 \dots$

$\rho_E = \to s_0 \to s_1 \to s_2 \to s_3 \to \dots$

$\rho_A = \to s_0 \to s'_1 \to s'_2 \to s'_3 \to \dots$

Automaton S

Starts at $\to s_0$, $\to s_0$

In round i:

1. Adam selects a_i
2. Eve selects $s_i \xrightarrow{a_i} s_{i+1}$
3. Adam selects $s'_i \xrightarrow{a_i} s'_{i+1}$

Winning condition for Eve: ρ_A is accepting $\Rightarrow \rho_E$ is accepting

(Fair) Simulation Game

$w = a_0 \quad a_1 \quad a_2 \quad a_3 \dots$

$\rho_A = \to p_0 \to p_1 \to p_2 \to p_3 \to \dots$

$\rho_E = \to s_0 \to s_1 \to s_2 \to s_3 \to \dots$

Automata I, S

Starts at $\to p_0$, $\to s_0$

In round i:

1. Adam selects a_i
2. Adam selects $p_i \xrightarrow{a_i} p_{i+1}$
3. Eve selects $s_i \xrightarrow{a_i} s_{i+1}$

Winning condition for Eve: ρ_A is accepting $\Rightarrow \rho_E$ is accepting

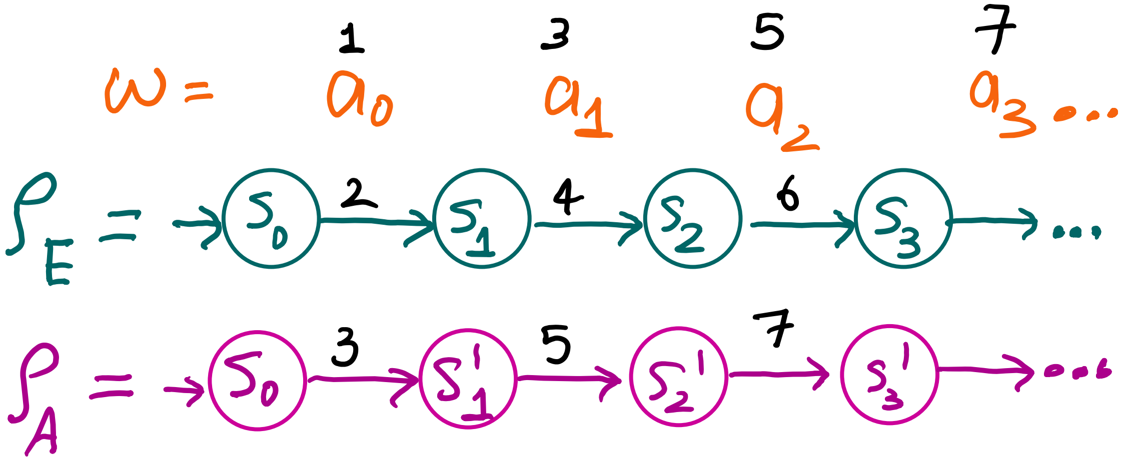
1-token game v/s Simulation game

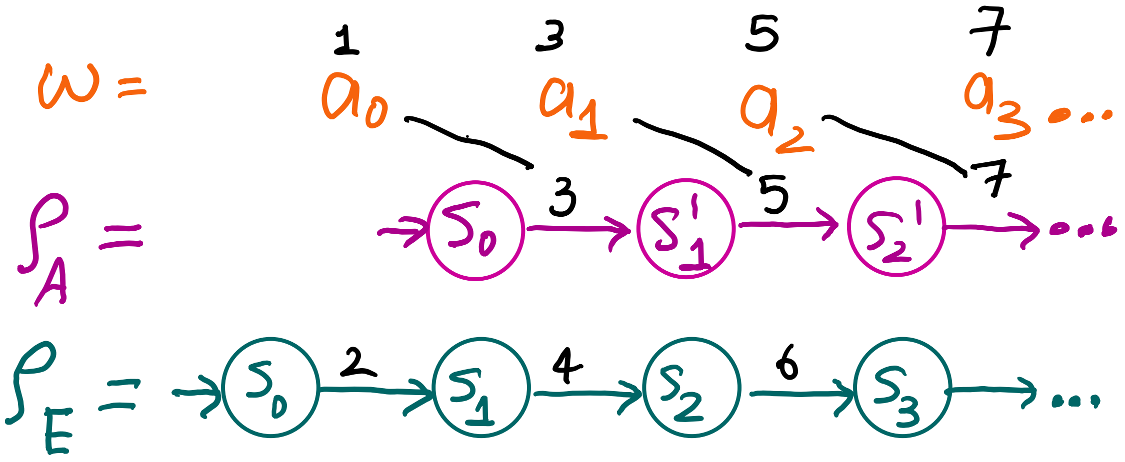
$$w = \overset{1}{a_0} \quad \overset{3}{a_1} \quad \overset{5}{a_2} \quad \overset{7}{a_3} \ldots$$

$$\rho_E = \to s_0 \xrightarrow{2} s_1 \xrightarrow{4} s_2 \xrightarrow{6} s_3 \to \ldots$$

$$\rho_A = \to s_0 \xrightarrow{3} s_1' \xrightarrow{5} s_2' \xrightarrow{7} s_3' \to \ldots$$

1-token game v/s Simulation game





1-token game v/s Simulation game

$$\omega = \overset{1}{a_0} \quad \overset{3}{a_1} \quad \overset{5}{a_2} \quad \overset{7}{a_3} \ldots$$

$$\rho_E = \rightarrow s_0 \xrightarrow{2} s_1 \xrightarrow{4} s_2 \xrightarrow{6} s_3 \rightarrow \ldots$$

$$\rho_A = \rightarrow s_0 \xrightarrow{3} s_1' \xrightarrow{5} s_2' \xrightarrow{7} s_3' \rightarrow \ldots$$

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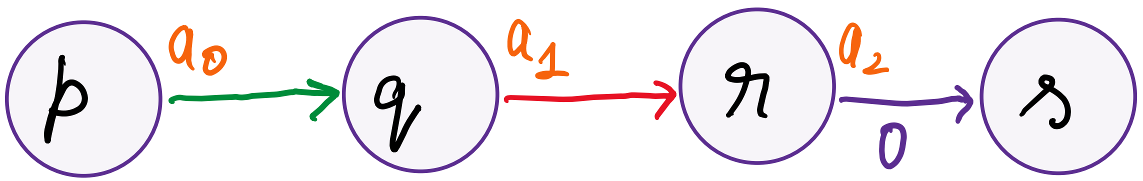
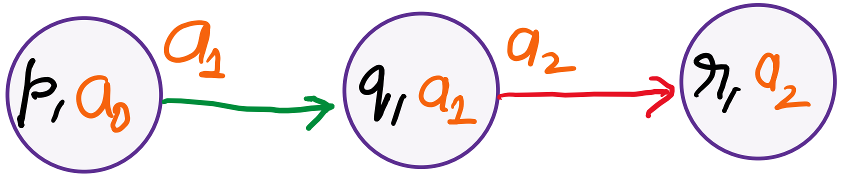
$$\rho_E = \rightarrow s_0 \xrightarrow{2} s_1 \xrightarrow{4} s_2 \xrightarrow{6} s_3 \rightarrow \ldots$$

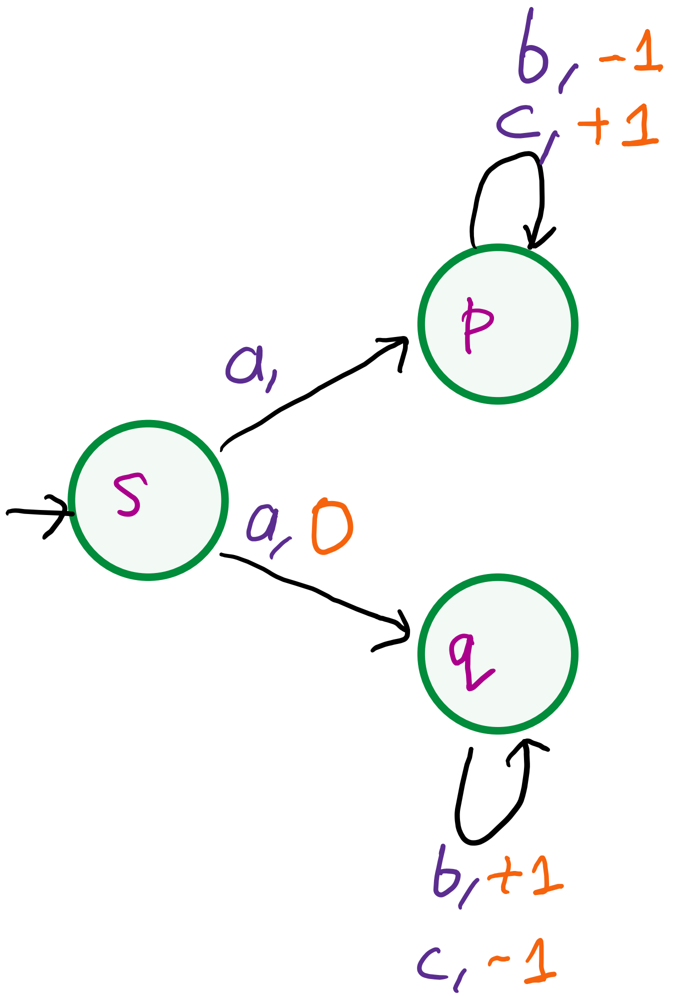


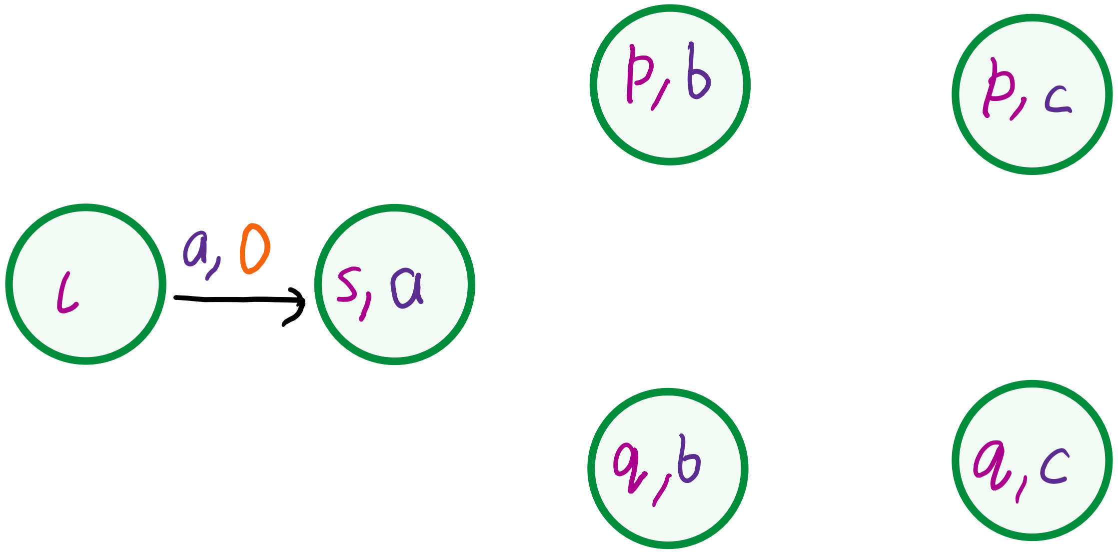
S



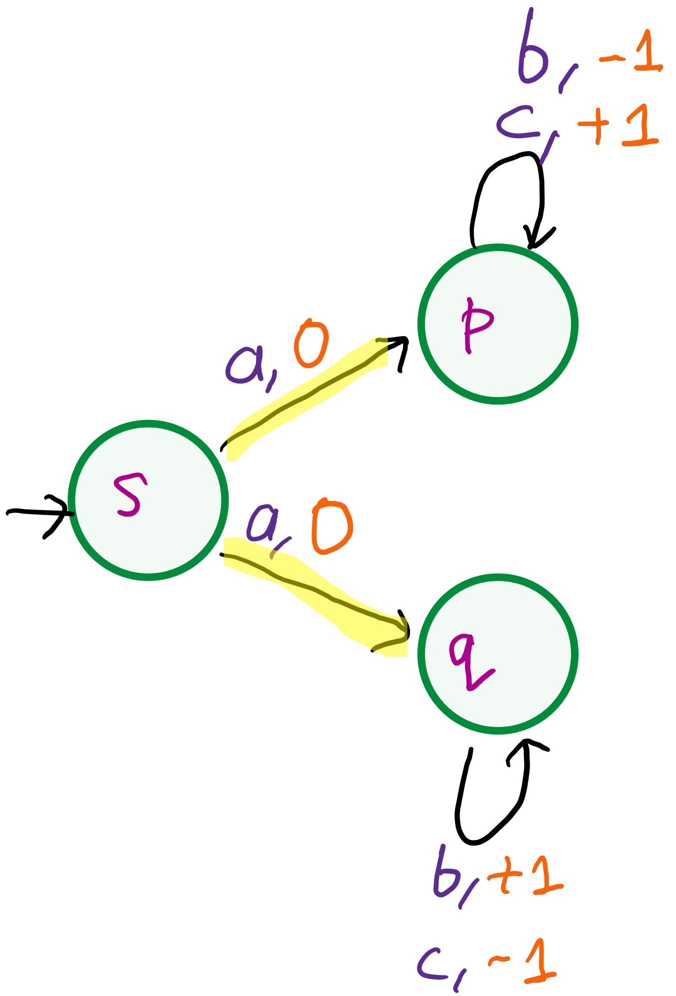
Delay (S)

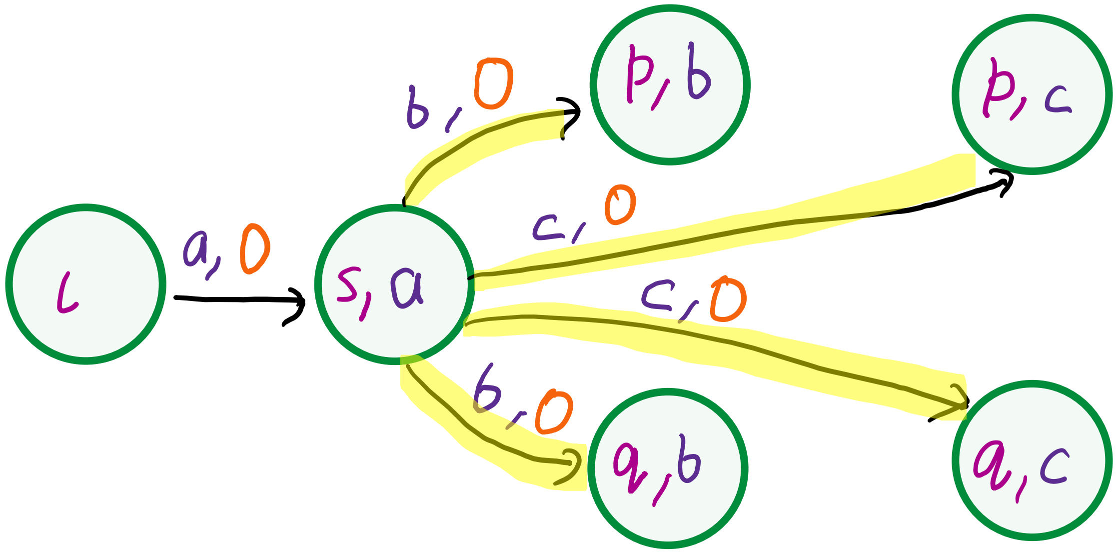
1-token game v/s Simulation game



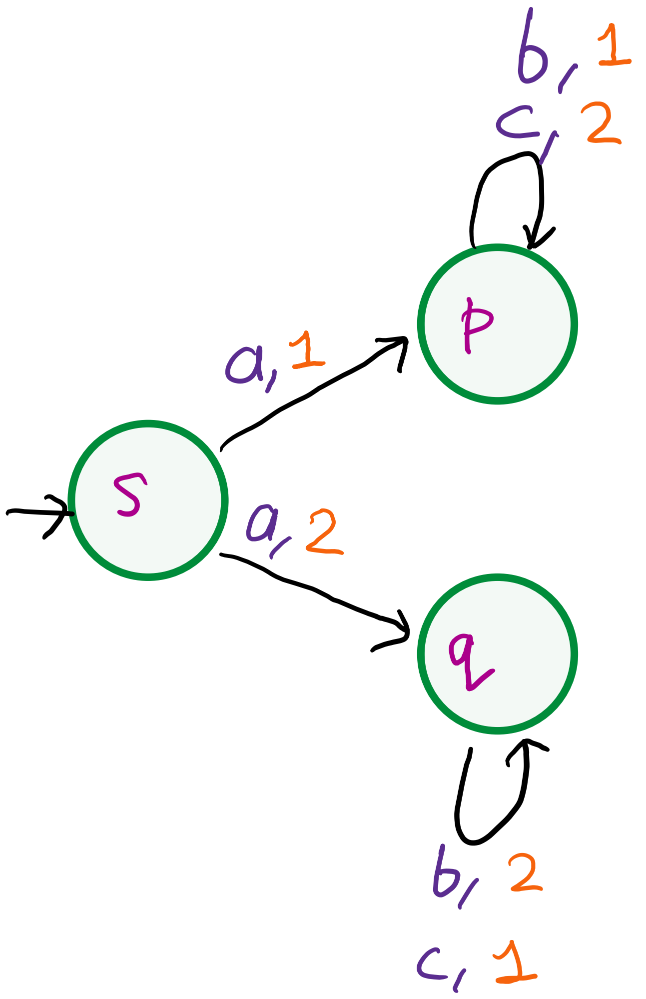


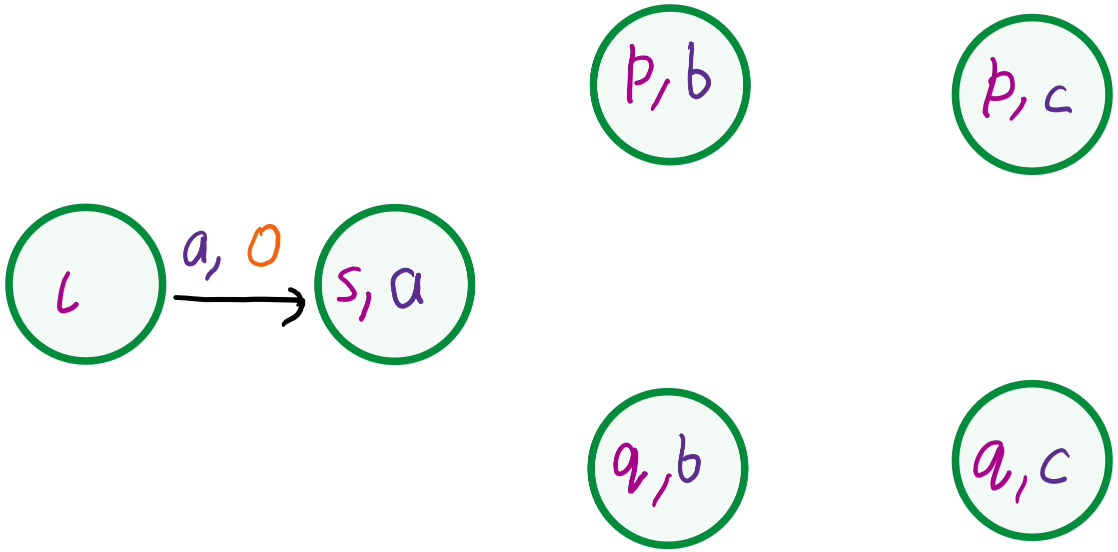
1-token game v/s Simulation game



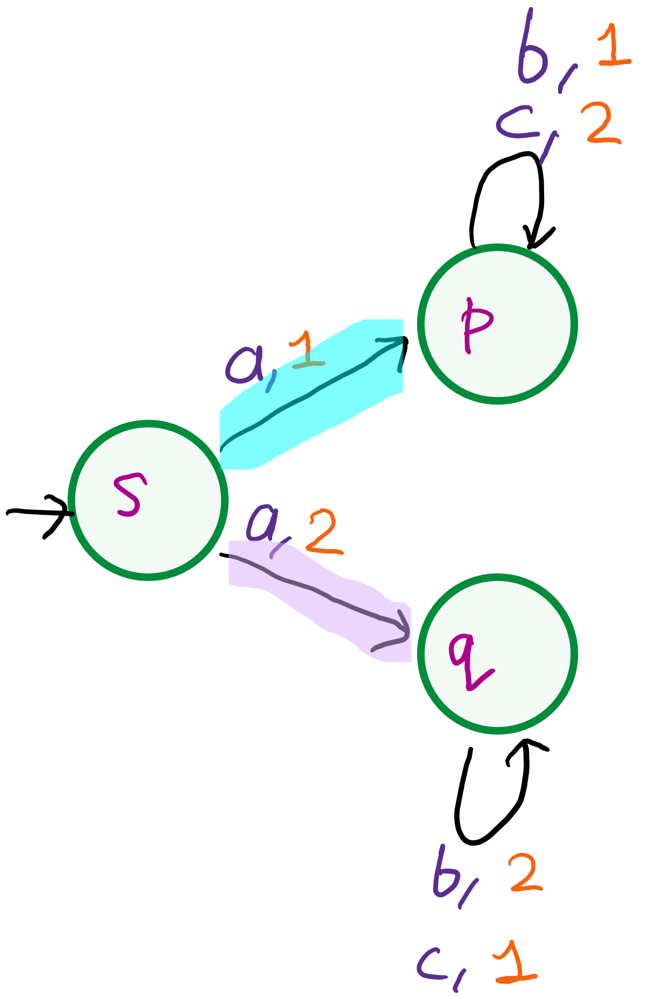


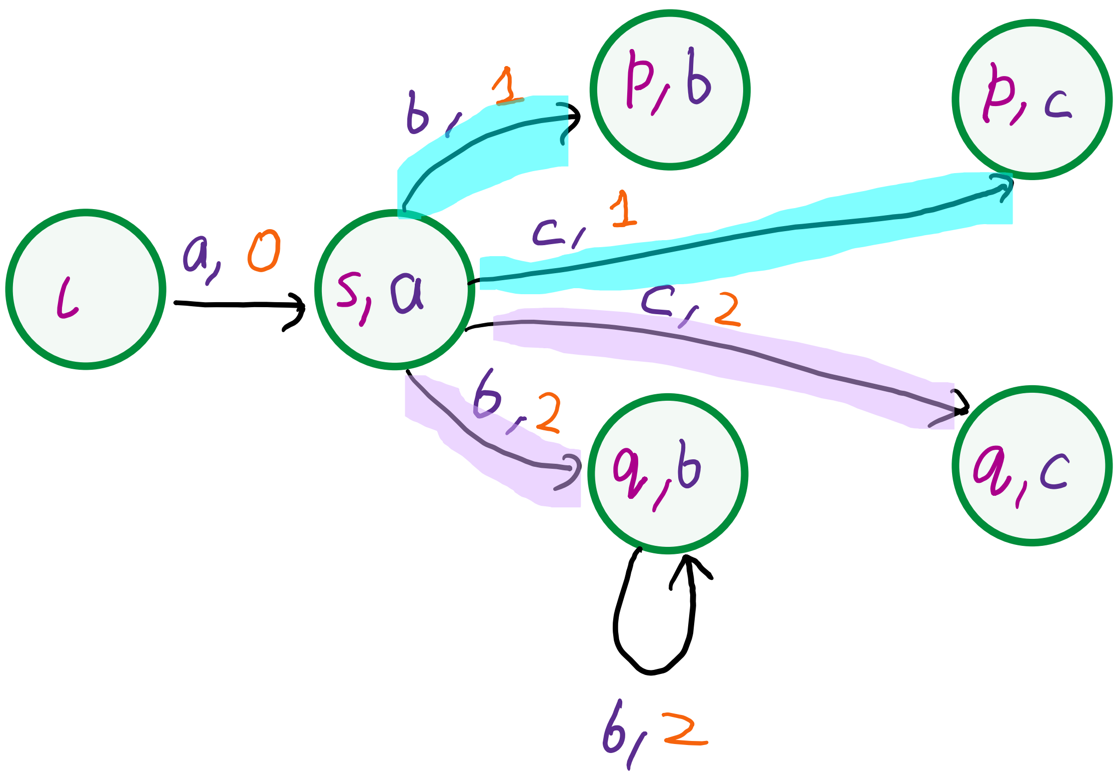
1-token game v/s Simulation game



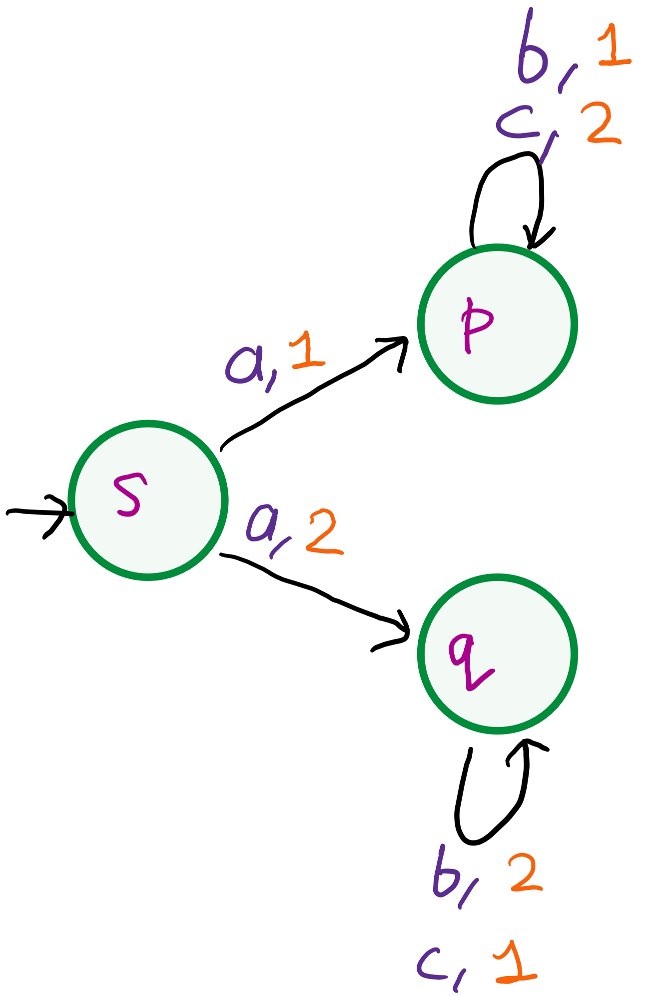


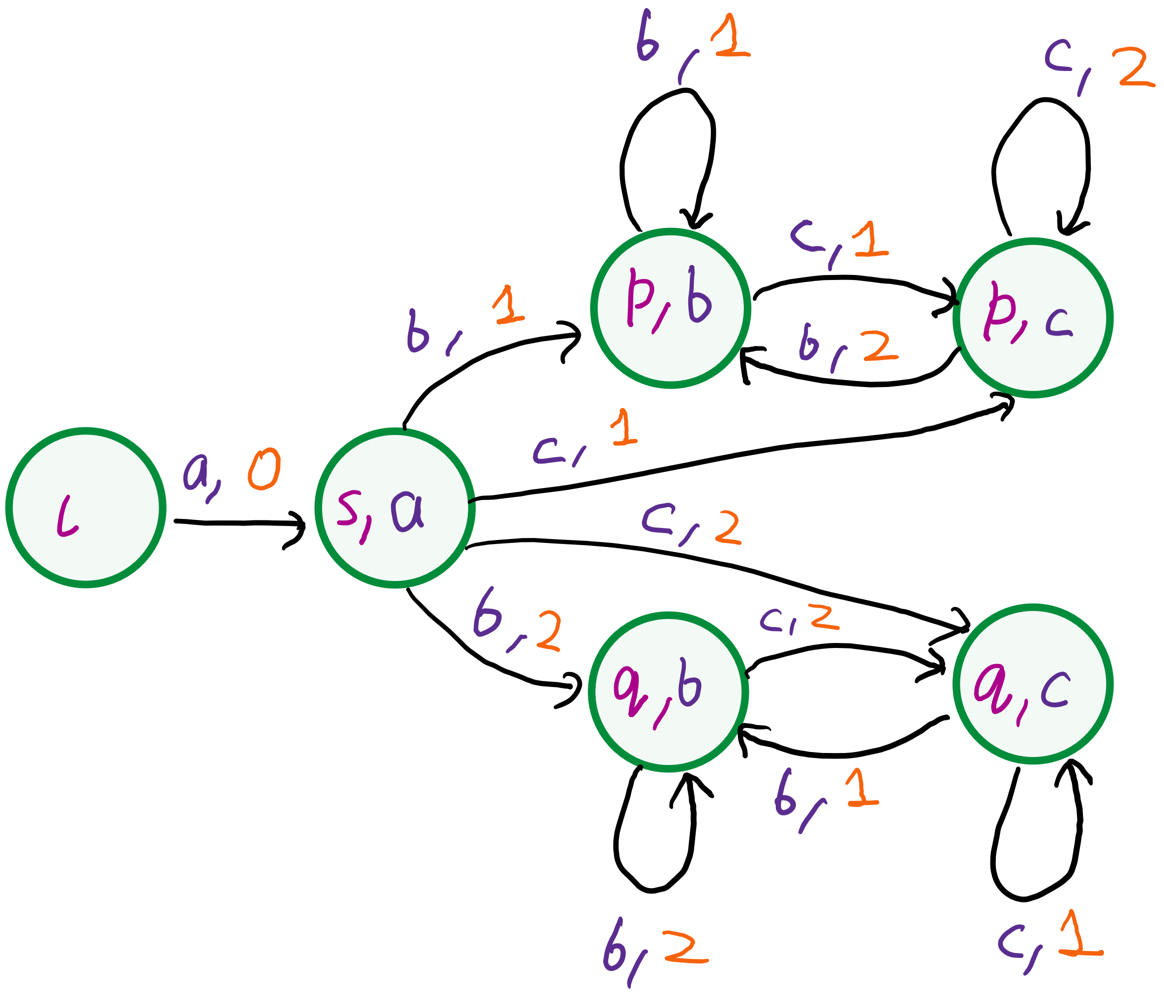
1-token game v/s Simulation game



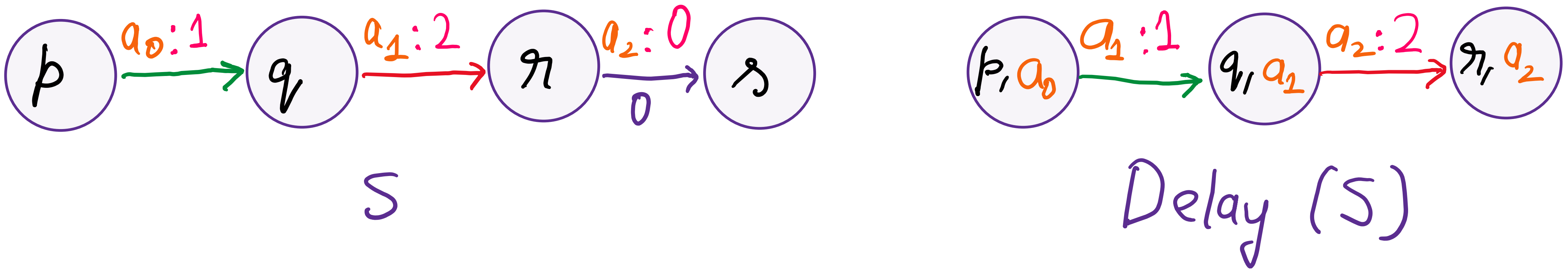


1-token game v/s Simulation game



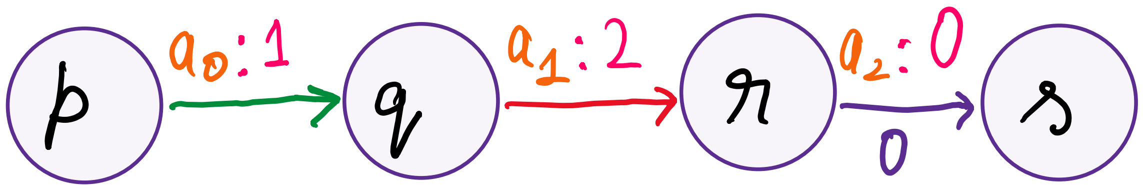


1-Token Game

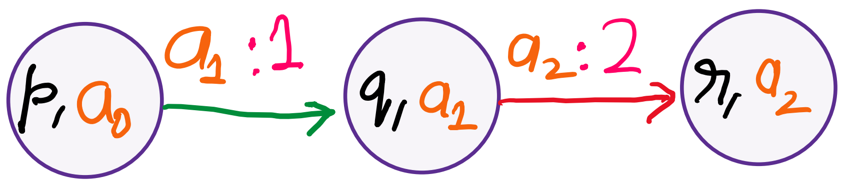


Lemma 1: Eve wins 1-token game on S iff S simulates Delay(S)

1-Token Game



S

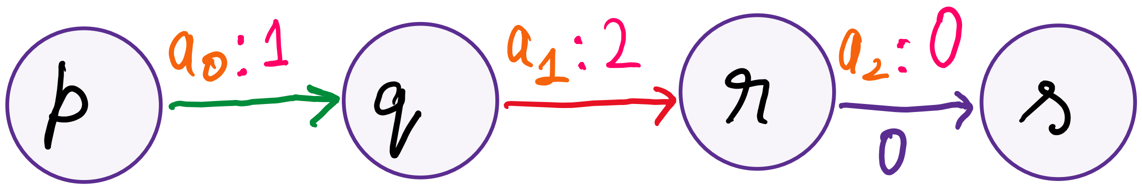


Delay (S)

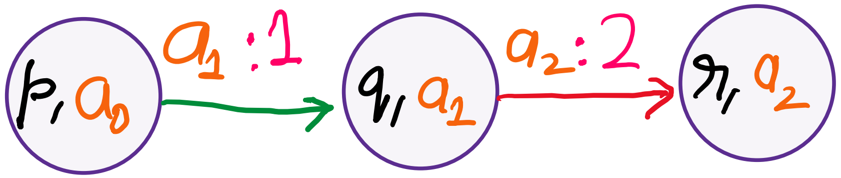
Lemma 1: Eve wins 1-token game on S iff S simulates Delay(S)

Lemma 2: Eve wins 1-token game on S $\Rightarrow$ Eve wins 1-token game on Delay(S)

1-Token Game



S



Delay (S)

Lemma 1: Eve wins 1-token game on S iff S simulates Delay(S)

Lemma 2: Eve wins 1-token game on S $\Rightarrow$ Eve wins 1-token game on Delay(S)

$$\mathrm{Delay}^2(S) \hookrightarrow \mathrm{Delay}(S) \hookrightarrow S$$

Lookahead Games

Theorem: Eve wins 1-token game on S

$$\Longrightarrow$$

S simulates $\mathrm{Delay}^k(S)$ for all $k \geq 0$

Lookahead Games

Theorem: Eve wins 1-token game on S

$$\Longrightarrow$$

$$\underbrace{S \text{ simulates } \mathrm{Delay}^k(S)}_{k\text{-lookahead game}} \text{ for all } k \geq 0$$

Lookahead Games

Theorem

On a semantically deterministic Büchi automaton $\mathcal{A}$,

Eve wins 1-token game on $\mathcal{A} \Leftrightarrow \mathcal{A}$ is H.D.

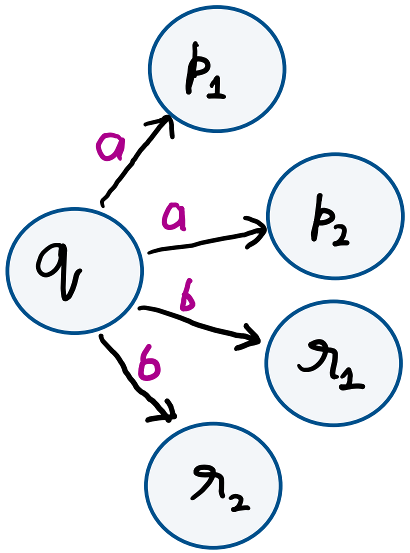
Lookahead Games

Theorem

On a semantically deterministic Büchi automaton A,

Eve wins 1-token game on $A \iff A$ is H.D.

Semantic determinism:



$$L(p_i) = a^{-1} \cdot L(q)$$

$$L(r_i) = b^{-1} \cdot L(q)$$

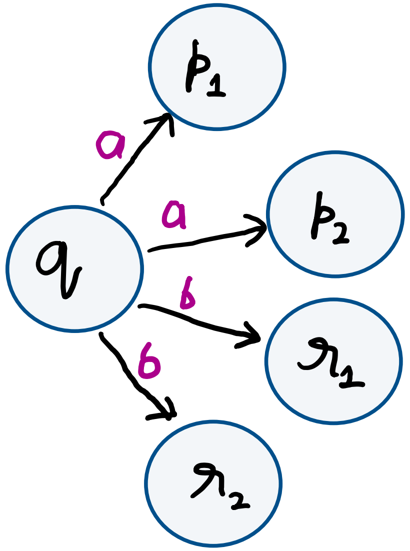
Lookahead Games

Theorem

On a semantically deterministic Büchi automaton A,

Eve wins 1-token game on $A \iff A$ is H.D.

Semantic determinism:



$L(p_i) = a^{-1} \cdot L(q)$

$L(r_i) = b^{-1} \cdot L(q)$

(Also known as residual automata)

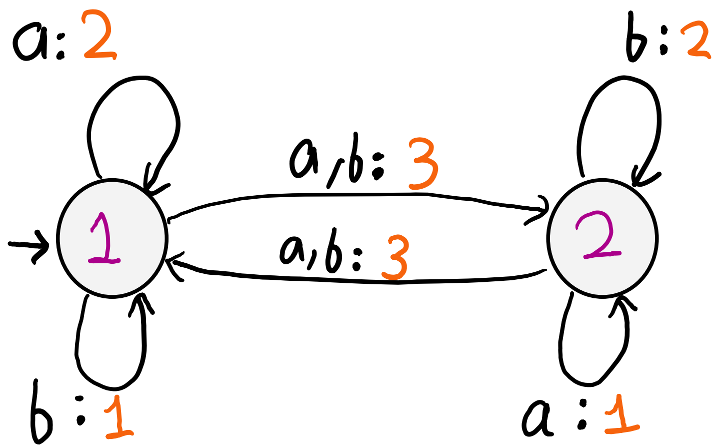
Lookahead Games

Theorem

On a semantically deterministic Büchi automaton A,

Eve wins 1-token game on $A \iff A$ is H.D.

Does not extend to parity automata:



Determinisation of H.D. Büchi Automata

Determinisation of H.D. Büchi Automata

Theorem [Kuperberg, Skrzypczak'15]

H.D. Büchi automaton with N states

$\Downarrow$

Deterministic Büchi automaton with $O(N^2)$ states

Determinisation of H.D. Büchi Automata

Theorem [Kuberberg, Skrzypczak'15]

H.D. Büchi automaton with N states

$\Downarrow$

Deterministic Büchi automaton with $O(N^2)$ states

Requires non-deterministic polynomial time

Problem: Can H.D. Büchi automata be determinised in polynomial time?

Determinisation of H.D. Büchi Automata

Theorem [Kuberberg, Skrzypczak '15]

H.D. Büchi automaton with N states

$\Downarrow$

Deterministic Büchi automaton with $O(N^2)$ states

Requires non-deterministic polynomial time

Problem: Can H.D. Büchi automata be determinised in polynomial time?

Theorem: Yes.

Determinisation of H.D. Büchi Automata

Theorem [Kuberberg, Skrzypczak '15]

H.D. Büchi automaton with N states

$\Downarrow$

Deterministic Büchi automaton with $O(N^2)$ states

Requires non-deterministic polynomial time

Problem: Can H.D. Büchi automata be determinised in polynomial time?

Theorem: Yes. Proof: Trim the 1-token game arena.

Conclusion

* H.D. Büchi automaton $\xrightarrow{\text{Poly. time}}$ Deterministic Büchi automaton

  N states $\leq N^2$ states

Open: Can we do better than N^2?

* Eve wins 1-token game $\Longleftrightarrow$ Eve wins k-lookahead game $\forall k \geq 0$.

Open: 2-token conjecture

Conclusion

* Checking history-determinism is NP-hard for parity automata.

Upper bound: EXPTIME

2-token conjecture $\Rightarrow$ PSPACE upper bound.

Bonus Fact

2-token conjecture: For any parity automaton A, Eve wins 2-token game on A iff A is HD.

$*$: For any parity automaton A with $L(A)=\Sigma^\omega$, Eve wins 2-token game on A iff A is HD.

Theorem: $* \implies$ 2-token conjecture.

Bonus Fact

2-token conjecture: For any parity automaton A with $L(A) = \Sigma^\omega$, Eve wins 2-token game on $A \iff A$ is HD.

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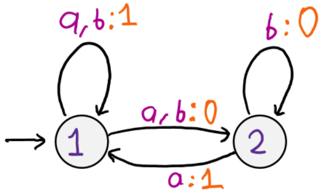
History-Determinism Game



Starts at $\to (1)$

Adam selects letter a_i

Eve selects transition $q_i \xrightarrow{a_i} q_{i+1}$

Winning cond^n for Eve:

Construct an accepting run if Adam's word is accepting.

HD Automata: Eve has a winning strategy

H.D. Game

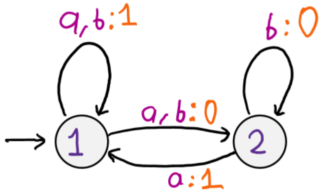
Adam	a	b	a	...
Eve	$1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \dots$			

Bonus Fact

2-token conjecture: For any parity automaton A with $L(A) = \Sigma^\omega$,

Eve wins 2-token game on A $\iff$ A is HD.

History-Determinism Game



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Adam	a	b	a	...
Eve	$1 \xrightarrow{a} 2$	$\xrightarrow{b} 2$	$\xrightarrow{a} 1$	$\rightarrow \ldots$

$\longrightarrow$ A parity game.

Joker Game

Kuperberg, Skrzypczak '15

Joker Game

Kuperberg, Skrzypczak '15

Automaton $\mathcal{S}$

Starts at $\to s_0$, $\to s_0$

In round i:

1. Adam selects a_i
2. Eve selects $s_i \xrightarrow{a_i} s_{i+1}$
3. Adam selects $s'_i \xrightarrow{a_i} s'_{i+1}$ OR $\underbrace{\text{selects } s_i \xrightarrow{a_i} s'_{i+1}}$ → Joker moves

$w = a_0 \quad a_1 \quad a_2 \quad a_3 \dots$

$\rho_E = \to s_0 \to s_1 \to s_2 \to s_3 \to \dots$

$\rho_A = \to s_0 \to s'_1 \quad s'_2 \to s'_3 \quad \dots$

Joker Game

Kuperberg, Skrzypczak '15

Automaton $\mathcal{S}$

Starts at $\to s_0$, $\to s_0$

$$\omega = \quad a_0 \quad a_1 \quad a_2 \quad a_3 \ldots$$

$$\rho_E = \to s_0 \to s_1 \to s_2 \to s_3 \to \ldots$$

$$\rho_A = \to s_0 \to s_1' \quad s_2' \to s_3' \quad \ldots$$

In round i:

1. Adam selects a_i
2. Eve selects $s_i \xrightarrow{a_i} s_{i+1}$
3. Adam selects $s_i' \xrightarrow{a_i} s_{i+1}'$ OR $\underbrace{\text{selects } s_i \xrightarrow{a_i} s_{i+1}'}$ → Joker moves

Eve's winning condition: If ρ_A is accepting and finitely many Jokers have been played, then ρ_E is accepting.

Joker Game

Theorem: For any Büchi automaton S, S is HD

$$\iff$$

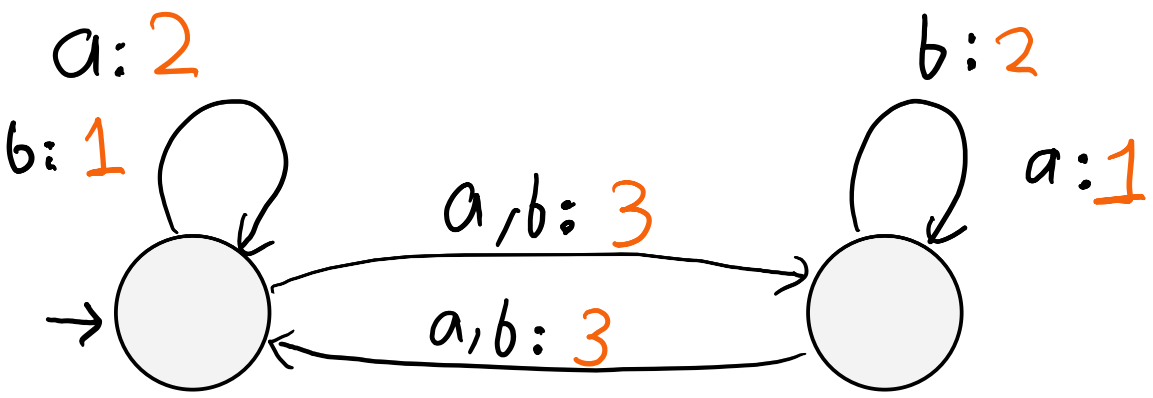
Eve wins Joker Game on S.

Joker Game

Theorem: For any Büchi automaton S, S is HD

$\iff$

Eve wins Joker Game on S.



Not true for parity automata.

III b. Determinisation

HD CoBüchi Automata

Theorem Kupferman'24

HD coBüchi automaton with n states $\rightsquigarrow$ Deterministic coBüchi automaton with 2^n states

HD CoBüchi Automata

Theorem Kupferman'24

HD coBüchi automaton with n states $\rightsquigarrow$ Deterministic coBüchi automaton with 2^n states

Tight.

Kuperberg, Skrzypczak'15

HD Büchi Automata

Theorem Kuperberg, Skrzypczak '15

HD Büchi automaton with n states $\rightsquigarrow$ Deterministic Büchi automaton with n^2 states

HD Büchi Automata

Theorem Kuperberg, Skrzypczak '15

HD Büchi automaton with n states $\rightsquigarrow$ Deterministic Büchi automaton with n^2 states

Non-det. Polynomial time.

Not known to be tight.

HD Büchi Automata

Theorem

HD Büchi automaton with n states $\rightsquigarrow$ Deterministic Büchi automaton with n^2 states

Polynomial Time

HD Büchi Automata

Theorem

HD Büchi automaton with n states $\rightsquigarrow$ Deterministic Büchi automaton with n^2 states

Polynomial Time

Proof Sketch. Modify and trim the arena of the Joker game.

Conclusion

* For $\mathcal{A}$ Büchi, Eve wins Joker Game $\Leftrightarrow$ $\mathcal{A}$ is H.D.

Conclusion

* For $\mathcal{A}$ Büchi, Eve wins Joker Game $\iff$ $\mathcal{A}$ is H.D.
* H.D. Büchi automaton $\xrightarrow{\text{Poly. time}}$ Deterministic Büchi automaton

  N states $\leq N^2$ states

Open: Can we do better than N^2 ?

Conclusion

* For $\mathcal{A}$ Büchi, Eve wins Joker Game $\Leftrightarrow$ $\mathcal{A}$ is H.D.

* H.D. Büchi automaton $\xrightarrow{\text{Poly. time}}$ Deterministic Büchi automaton
  N states $\qquad\qquad\qquad\qquad$ $\leq N^2$ states

Open: Can we do better than N^2?

* Recognising HD Parity Automata: **NP-hard**
  2-token conjecture $\Longrightarrow$ **PSPACE** upper bound.