

# Joker Games for checking History-Determinism

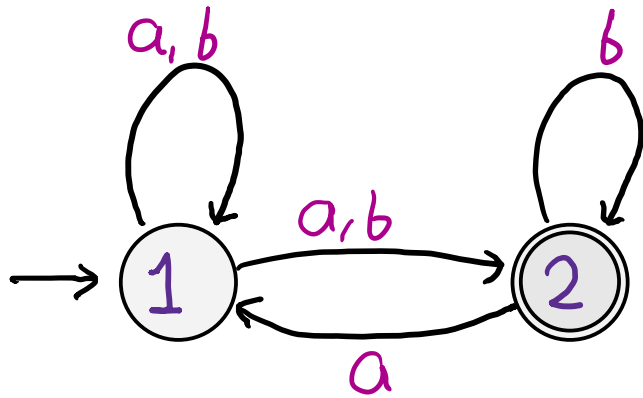
Aditya Prakash

University of Warwick, UK

Ongoing work with Udi Boker, Marcin Jurdziński, Karoliina Lehtinen

# Running Example

Büchi Automata

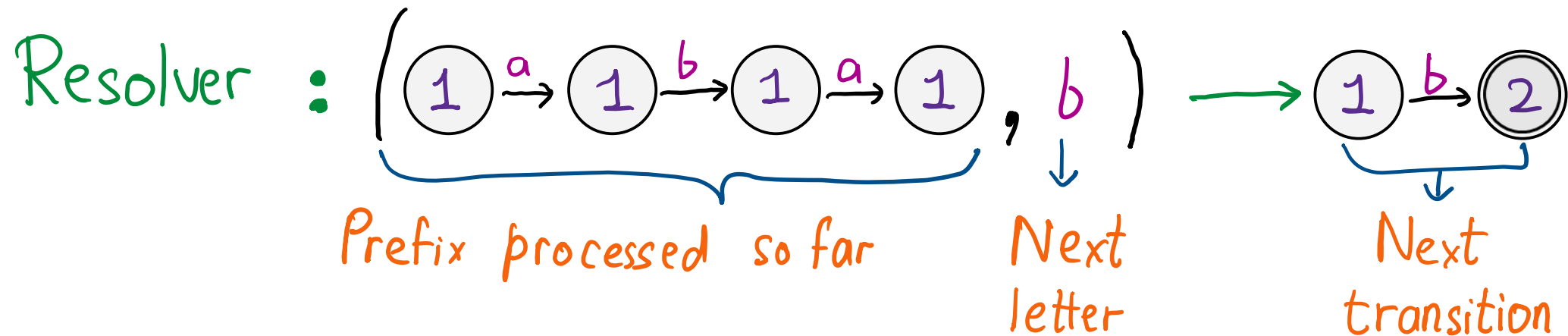


Language : Words that see  $b$  only often.

# History-Deterministic Automata

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A non-deterministic automaton is history-deterministic if non-deterministic choices can be resolved 'on-the-fly' while processing the input.

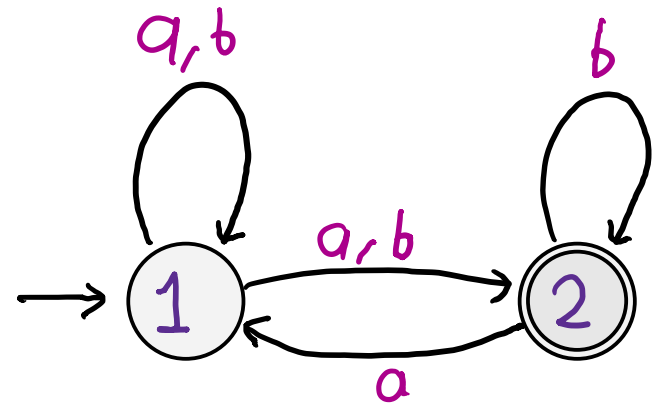


# History-Determinism Game

Starts at  $\rightarrow 1$

Adam selects letter  $a_i$

Eve selects transition  $q_i \xrightarrow{a_i} q_{i+1}$

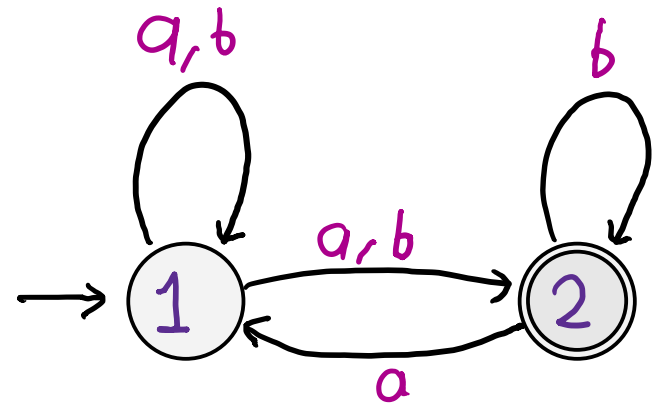


# History-Determinism Game

Starts at  $\rightarrow$  ①

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H.D. Game

Adam

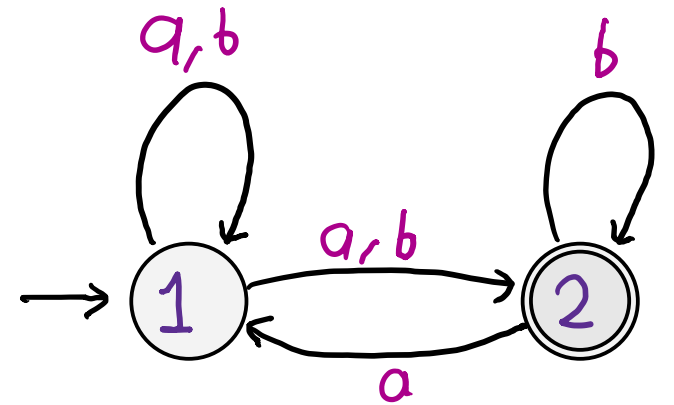
Eve ①

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H.D. Game

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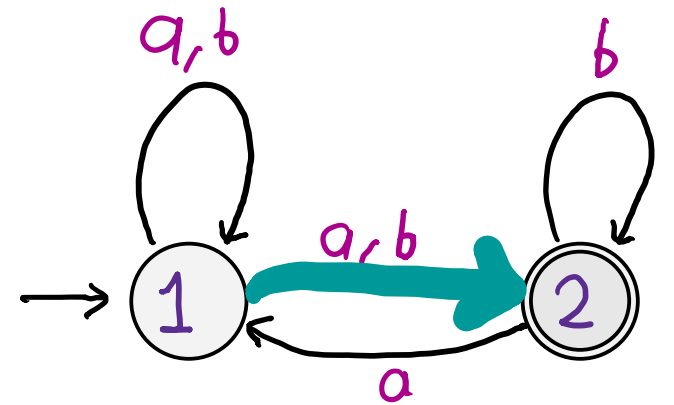
Eve  $1$

# History-Determinism Game

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H.D. Game

Adam  $a$

Eve  $1 \xrightarrow{a} 2$

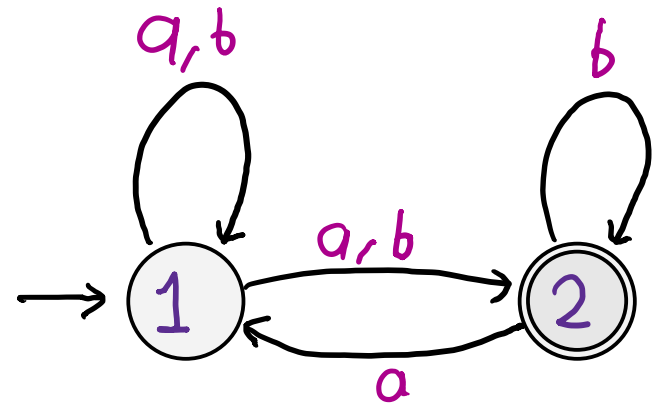


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H.D. Game

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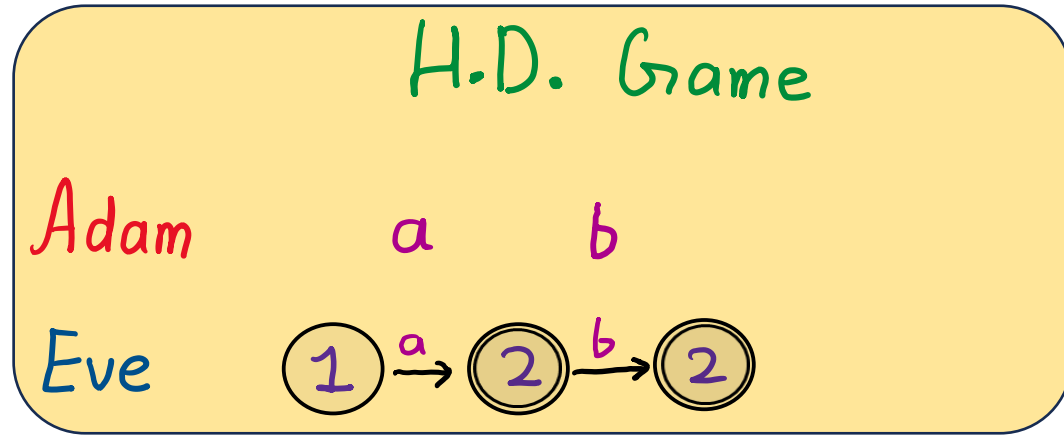
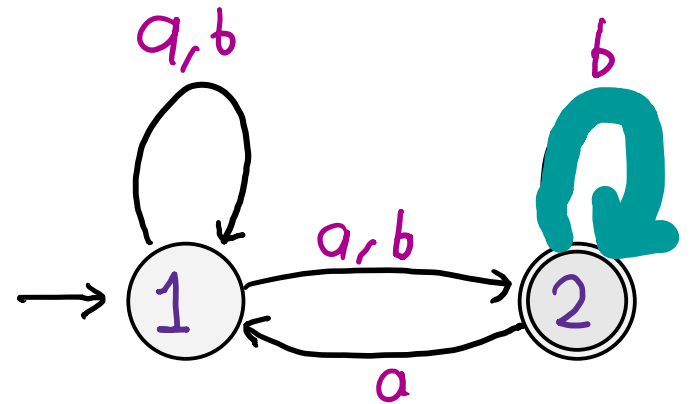
Eve      1  $\xrightarrow{a}$  2

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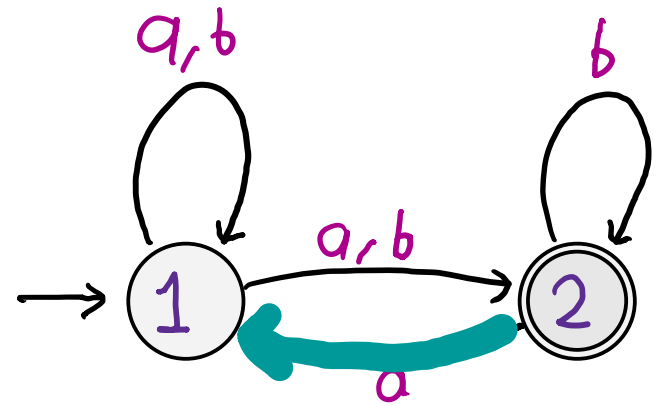


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H.D. Game

Adam a b a ...

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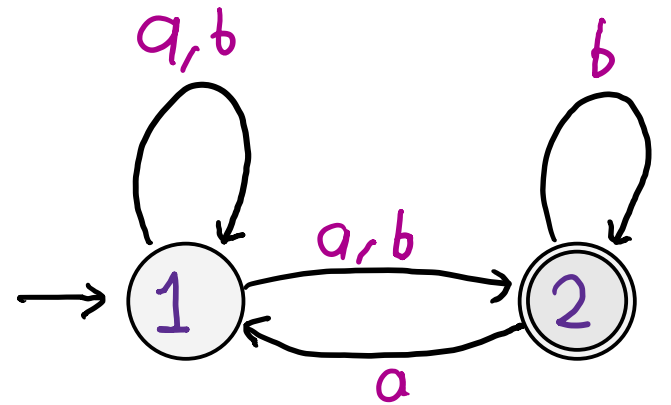
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Construct an accepting run whenever Adam's word is accepting.



H.D. Game

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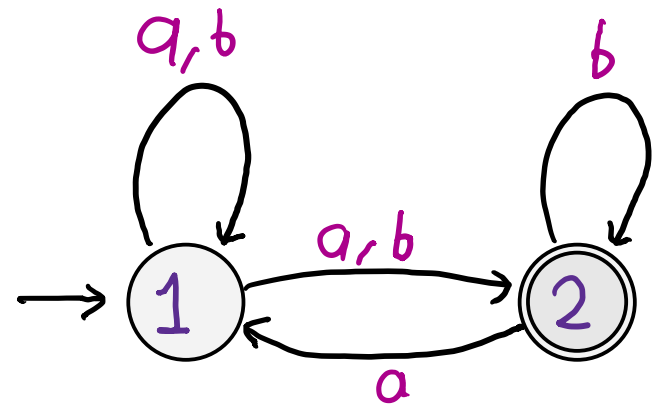
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HD Automata: Eve has a winning strategy



H.D. Game

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Eve  $1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \dots$

# Determinisation of H.D. Büchi Automata

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Theorem [Kuperberg, Skrzypczak '15]

H.D. Büchi automaton with  $N$  states

⇓

Deterministic Büchi automaton with  $\Theta(N^2)$  states

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Requires non-deterministic polynomial time

**Problem:** Can H.D. Büchi automata be determinised in polynomial time?



# Determinisation of H.D. Büchi Automata

Theorem [Kuperberg, Skrzypczak '15]

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$\Downarrow$

Deterministic Büchi automaton with  $\Theta(N^2)$  states

Requires non-deterministic polynomial time

**Problem:** Can H.D. Büchi automata be determinised in polynomial time?

**Theorem:** Yes

# Complexity of checking History-Determinism

## History-Determinism Game

Starts at  $\rightarrow 1$

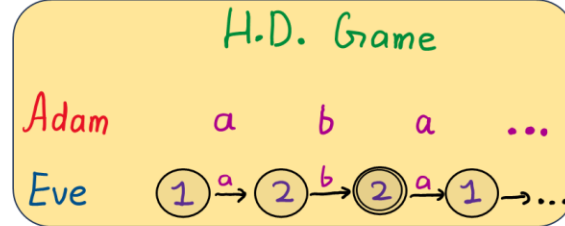
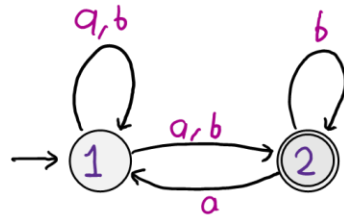
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HD Automata: Eve has a winning strategy



Solving HD game: EXPTIME

[Henzinger, Piterman '06]

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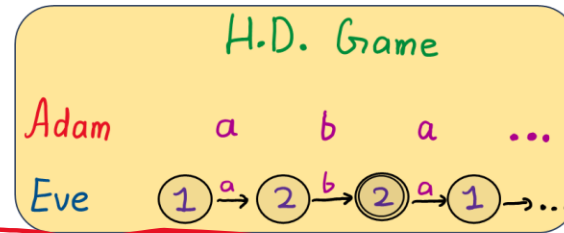
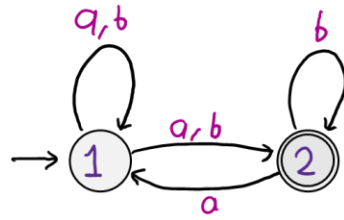
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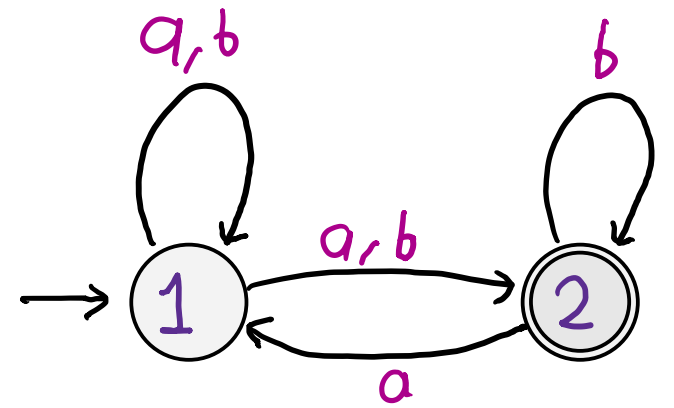


Solving HD game: EXPTIME

[Henzinger, Piterman '06]

Determinisation is expensive!

# 2-Token Games



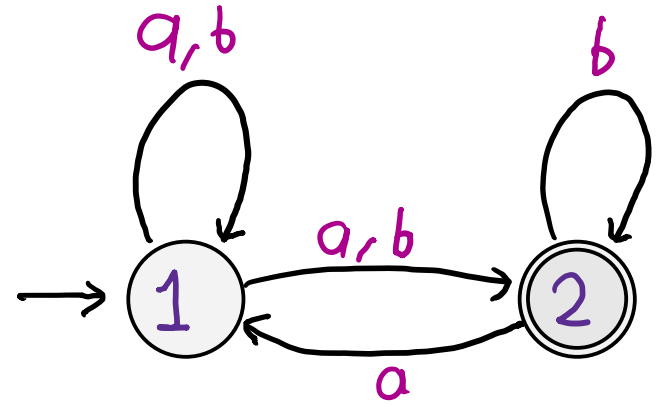
# 2-Token Games

Starts at  $\rightarrow$  (1),  $\rightarrow$  (1),  $\rightarrow$  (1)

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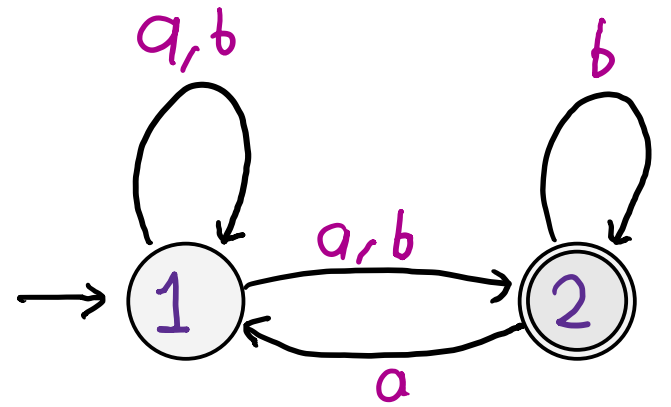
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## 2-Token Game

Adam

Eve (1)

Adam (1)

Adam (1)

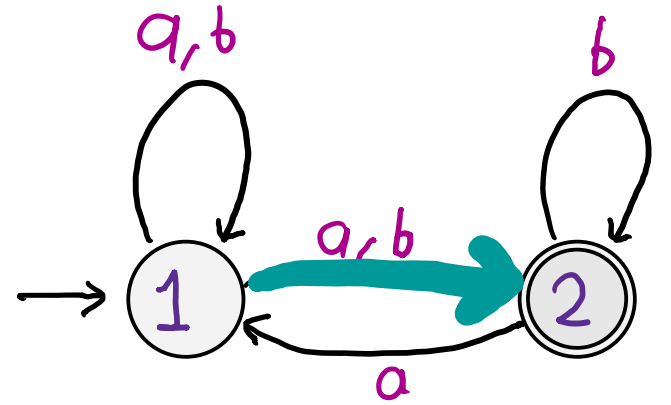
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 $p_i^2 \xrightarrow{a_i} p_{i+1}^2$



## 2-Token Game

Adam a

Eve (1)  $\xrightarrow{a}$  ((2))

Adam (1)

Adam (1)

# 2-Token Games

Starts at  $\rightarrow$  (1),  $\rightarrow$  (1),  $\rightarrow$  (1)

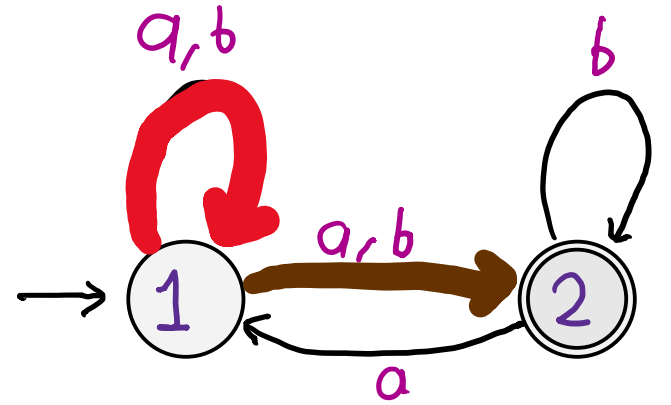
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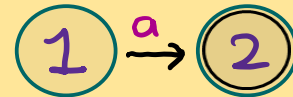


## 2-Token Game

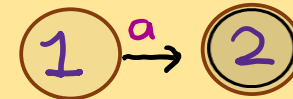
Adam

a

Eve



Adam



Adam





# 2-Token Games

Starts at  $\rightarrow$  (1),  $\rightarrow$  (1),  $\rightarrow$  (1)

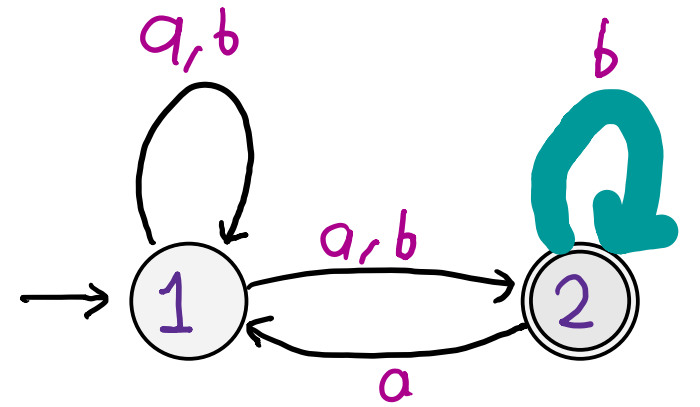
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$p_i^2 \xrightarrow{a_i} p_{i+1}^2$



## 2-Token Game

Adam            a            b

Eve            (1)  $\xrightarrow{a}$  (2)  $\xrightarrow{b}$  (2)

Adam            (1)  $\xrightarrow{a}$  (2)

Adam            (1)  $\xrightarrow{a}$  (1)

# 2-Token Games

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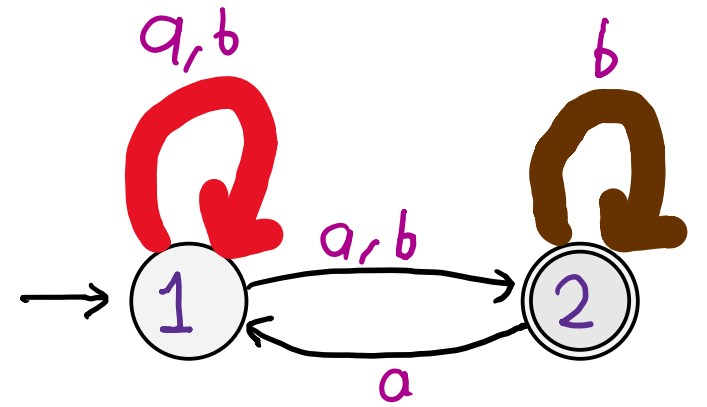
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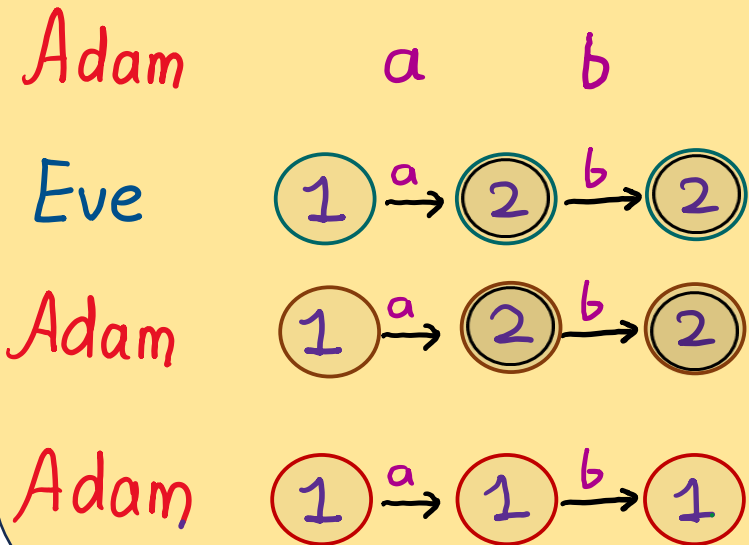
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## 2-Token Game



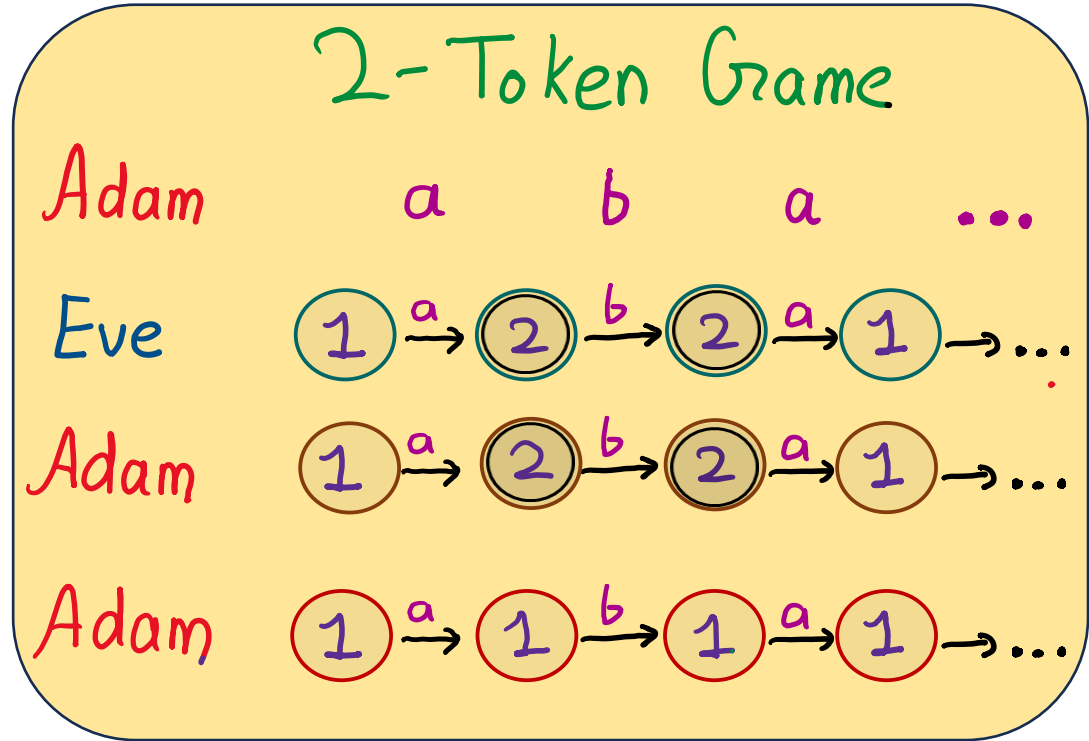
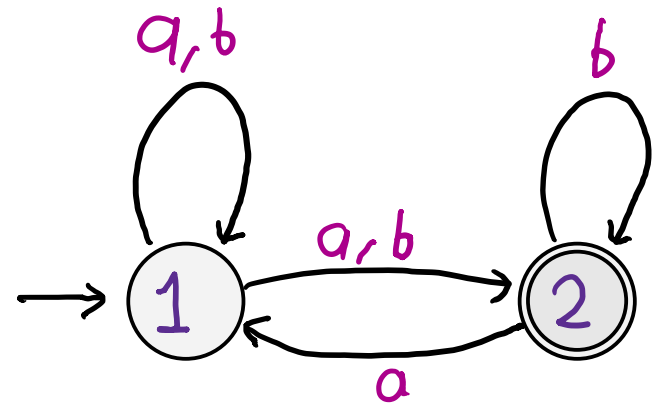
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Adam selects transitions

$$\begin{array}{c} p_i^2 \xrightarrow{a_i} p_{i+1}^1 \\ p_i^1 \xrightarrow{a_i} p_{i+1}^2 \end{array}$$


# 2-Token Games

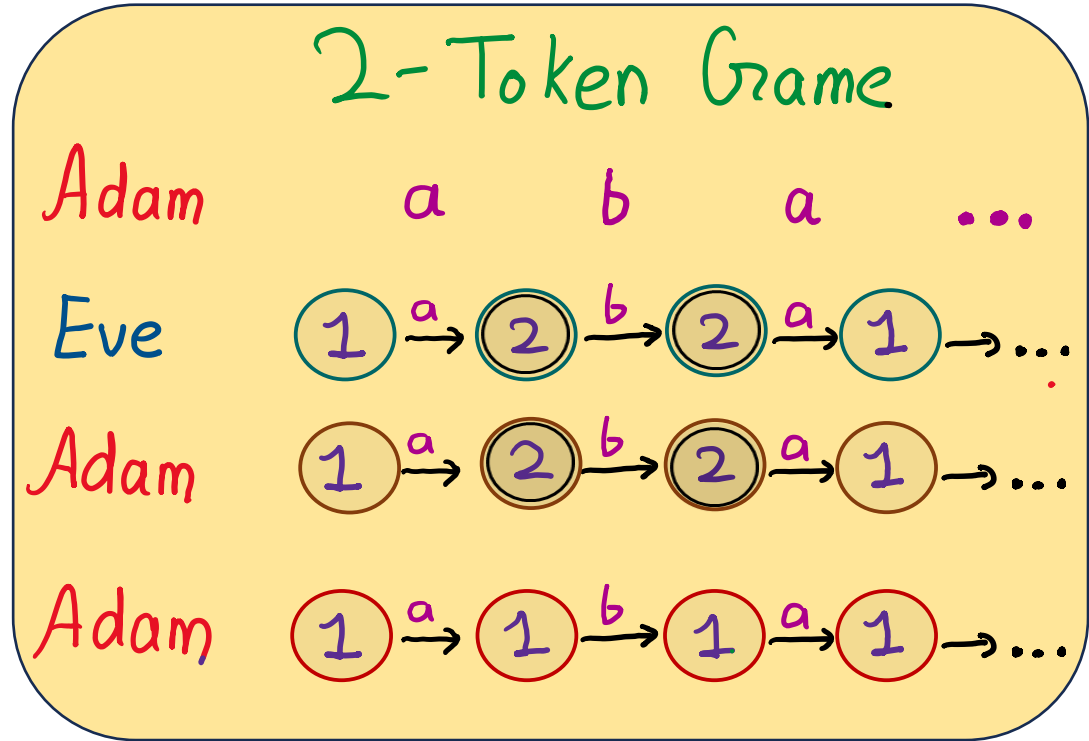
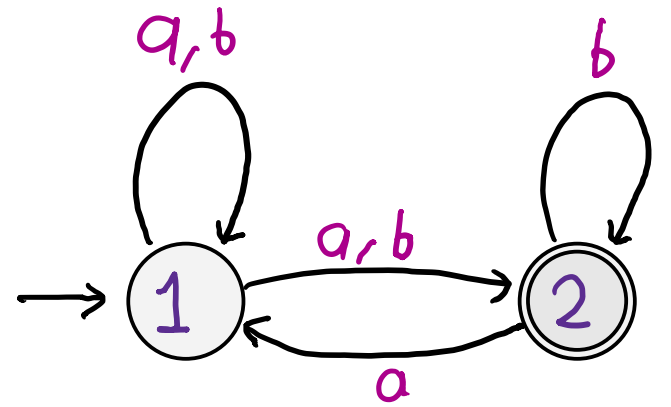
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Winning cond<sup>n</sup>. for Eve: Construct an accepting run if one of Adam's run is accepting.



# 2-Token Games vs. HD Game

## 2-Token Games

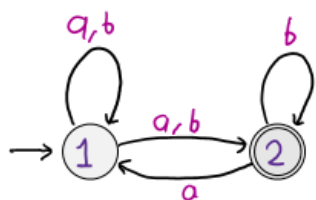
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### 2-Token Game

Adam a b a ...

Eve  $1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \dots$

Adam  $1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \dots$

Adam  $1 \xrightarrow{a} 1 \xrightarrow{b} 1 \xrightarrow{a} 1 \rightarrow \dots$

## History-Determinism Game

Starts at  $\rightarrow 1$

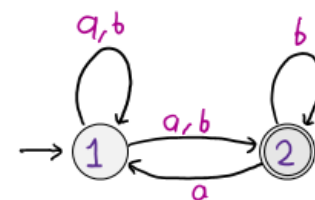
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HD Automata: Eve has a winning strategy



### H.D. Game

Adam a b a ...

Eve  $1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \dots$

# 2-Token Games vs. HD Game

## 2-Token Games

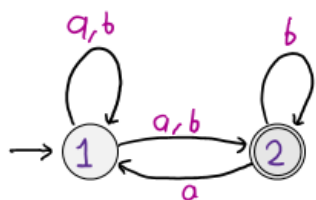
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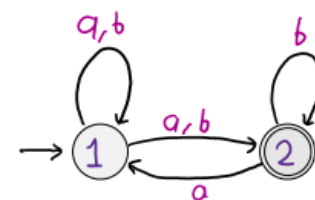
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# 2-Token Games vs. HD Game

## 2-Token Games

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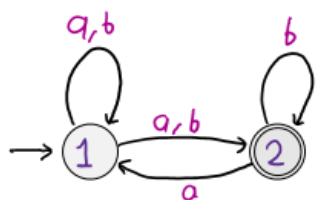
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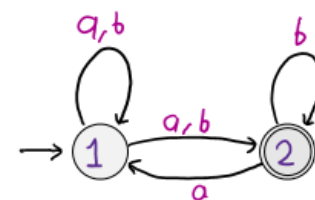
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### H.D. Game

Adam a b a ...

Eve  $1 \xrightarrow{a} 2 \xrightarrow{b} 2 \xrightarrow{a} 1 \rightarrow \dots$

Eve wins H.D. game  $\Rightarrow$  Eve wins 2-token game

# Token Games for checking History-Determinism

Lemma [Bagnol, Kuperberg '18]

Eve wins 2-token game  $\iff$  Eve wins  $k$ -token game for all  $k \geq 0$ .



# Token Games for checking History-Determinism

Lemma [Bagnol, Kuperberg '18]

Eve wins 2-token game  $\iff$  Eve wins  $k$ -token game for all  $k \geq 0$ .

Theorem [Bagnol, Kuperberg '18]

For all Büchi automata  $A$

Eve wins 2-token game on  $A \iff A$  is H.D.

# Token Games for checking History-Determinism

Theorem [Boker, Kuperberg, Lehtinen, Skrzypczak '20]

For all co-Büchi automata  $A$ ,

Eve wins 2-token game on  $A \Leftrightarrow A$  is H.D.

# Token Games for checking History-Determinism

Theorem [Boker, Kuperberg, Lehtinen, Skrzypczak '18]

For all co-Büchi automata  $A$ ,  
Eve wins 2-token game on  $A \Leftrightarrow A$  is H.D.

2-Token Conjecture: For all parity automata  $A$ ,  
Eve wins 2-token game on  $A \Leftrightarrow A$  is H.D.

Token Games for checking History-Determinism

Theorem

# Token Games for checking History-Determinism

## Theorem

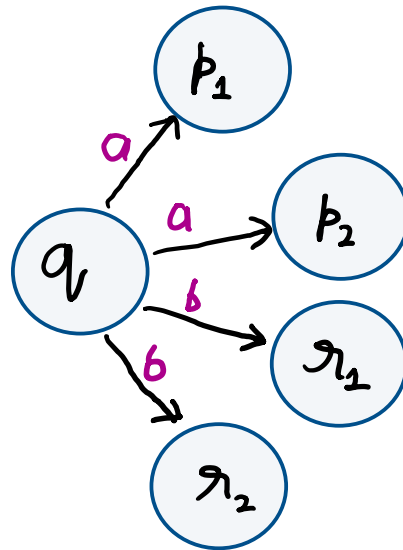
On a semantically deterministic Büchi automaton  $A$ ,  
Eve wins 1-token game on  $A \iff A$  is H.D.

# Token Games for checking History-Determinism

## Theorem

On a semantically deterministic Büchi automaton  $A$ ,  
Eve wins 1-token game on  $A \iff A$  is H.D.

Semantic determinism:



$$L(p_i) = a^{-1} \cdot L(q_v)$$

$$L(r_i) = b^{-1} \cdot L(q_v)$$

# Token Games for checking History-Determinism

## Theorem

On a semantically deterministic Büchi automaton  $A$ ,  
Eve wins 1-token game on  $A \iff A$  is H.D.

Corollary, also [Bagnol and Kuperberg' 18]

For all Büchi automata  $A$

Eve wins 2-token game on  $A \iff A$  is H.D.

# Token Games for checking History-Determinism

## One-Token Conjecture:

On a semantically deterministic parity automaton  $A$ ,

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# Token Games for checking History-Determinism

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On a semantically deterministic parity automaton  $A$ ,

Eve wins 1-token game on  $A \iff A$  is H.D.

**Observation:** One-token conjecture  $\implies$  Two-token conjecture

# Conclusion

\*

	Büchi	co-Büchi	Parity
1-token conjecture	[To appear]	Open	Open
2-token conjecture	[BK'18]	[BKLS'20]	Open

# Conclusion

*		Büchi	co-Büchi	Parity
	1-token conjecture	[To appear]	Open	Open
	2-token conjecture	[BK'18]	[BKLS'20]	Open
*	For $A$ Büchi, Eve wins Joker Game $\Leftrightarrow$	$A$ is H.D.		

# Conclusion

- \*

	Büchi	co-Büchi	Parity
1-token conjecture	[To appear]	Open	Open
2-token conjecture	[BK'18]	[BKLS'20]	Open
- \* For  $A$  Büchi, Eve wins Joker Game  $\Leftrightarrow A$  is H.D.
- \* H.D. Büchi automaton  $\xrightarrow{\text{Poly. time}}$  Deterministic Büchi automaton  
 $N$  states  $\subseteq N^2$  states

# Conclusion

- \*

	Büchi	co-Büchi	Parity
1-token conjecture	[To appear]	Open	Open
2-token conjecture	[BK'18]	[BKLS'20]	Open
- \* For  $A$  Büchi, Eve wins Joker Game  $\Leftrightarrow A$  is H.D.
- \* H.D. Büchi automaton  $\xrightarrow{\text{Poly. time}}$  Deterministic Büchi automaton  
 $N$  states  $\subseteq N^2$  states
- \* Open: Can we do better than  $N^2$ ?

# Summary

→ 2-token conjecture for H.D. Büchi

→ Determinisation of H.D. Büchi in  $P$

→ 1-token conjecture for H.D. Büchi

Henzinger,  
Piterman, 2006

Bagnol,  
Kuperberg, 2018

Boker, Jurdziński,  
Lehtinen & me!, to appear

Kuperberg,  
Skrzypczak, 2015

Boker, Kuperberg,  
Lehtinen, Skrzypczak, 2020

→ Determinisation of HD Büchi in  $NP$

→ 2-token conjecture for co-Büchi

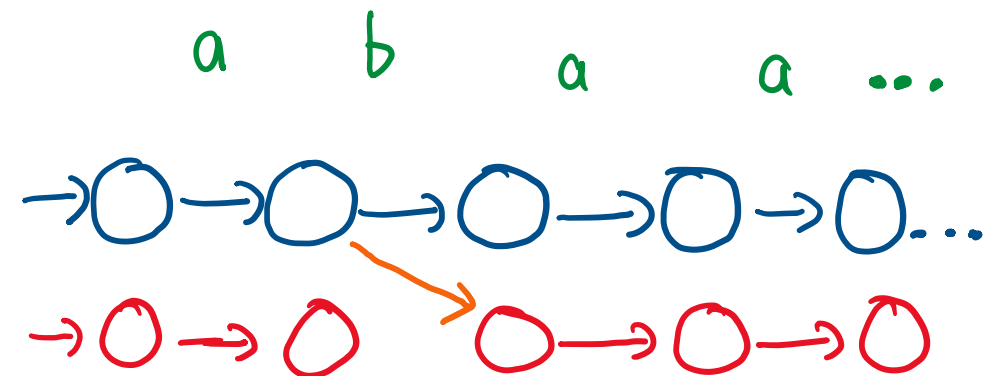
→ Deciding HDness of co-Büchi in  $P$

Bonus

Slides

# Joker Games

Starts at  $\rightarrow q_0 \rightarrow q_0$



1. Adam selects letter  $a_i$

2. Eve selects transition  $q_i \xrightarrow{a_i} q_{i+1}$

3. Adam selects transition  $p_i \rightarrow p_{i+1}$  or plays **Joker**

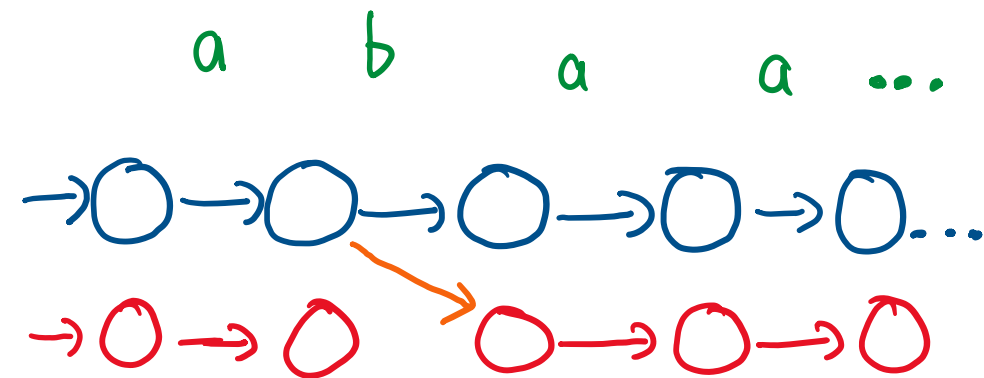
and selects transition  $q_i \rightarrow p_{i+1}$

**Winning Condition for Eve:** Eve constructs an accepting run if Adam's 'run' is accepting and Adam has played finite Jokers.



# 1-Joker Games

Starts at  $\rightarrow q_0 \rightarrow q_0$



1. Adam selects letter  $a_i$

2. Eve selects transition  $q_i \xrightarrow{a_i} q_{i+1}$

3. Adam selects transition  $p_i \rightarrow p_{i+1}$  or plays Joker

and selects transition  $q_i \rightarrow p_{i+1}$

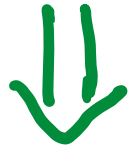
Winning Condition for Eve: Eve constructs an accepting run if Adam's 'run' is accepting and Adam has played  $\leq 1$  Joker.

For a Büchi Automaton:

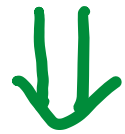
→ Running time for solving 1-Joker game:  $\Theta(n^3 m |\Sigma|^2)$

→ Running time for solving 2-Token game:  $\Theta(n^4 m^2 |\Sigma|^2)$

1 - Token Conjecture



1 - Joker Conjecture



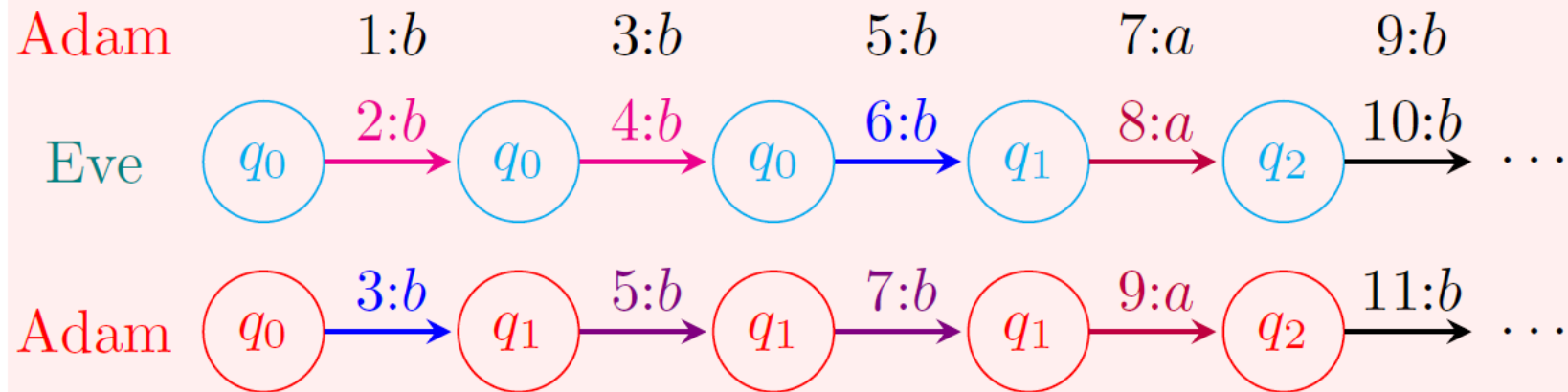
Joker Conjecture



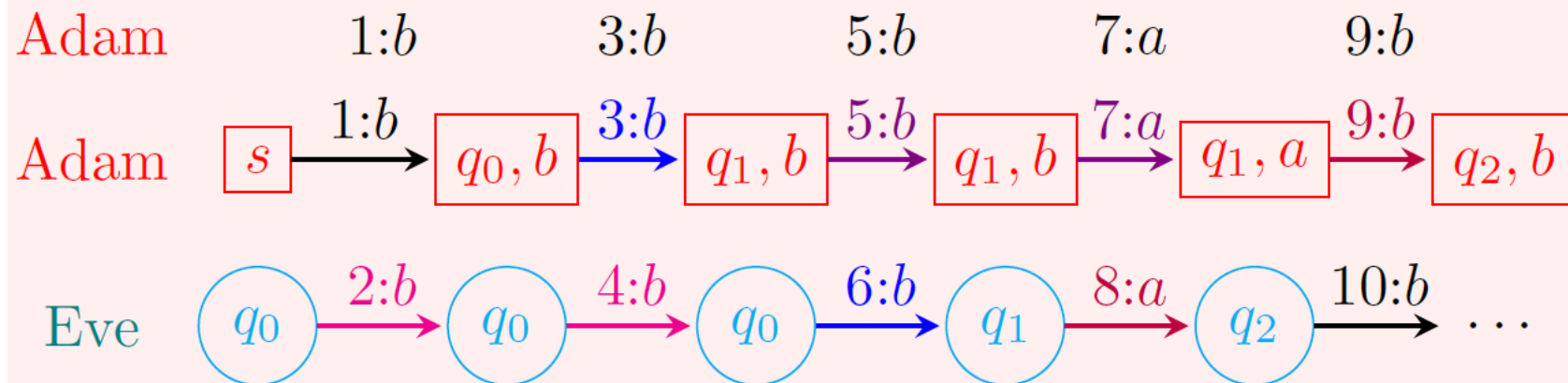
2 - Token Conjecture

# Power of 1-Token Game

## One-Token Game

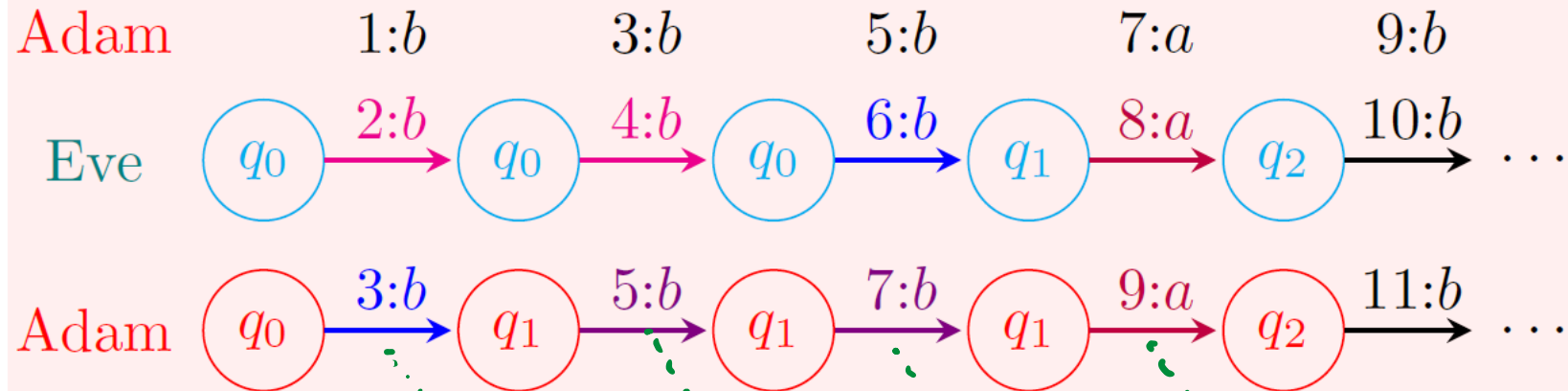


## Simulating 1-lookahead

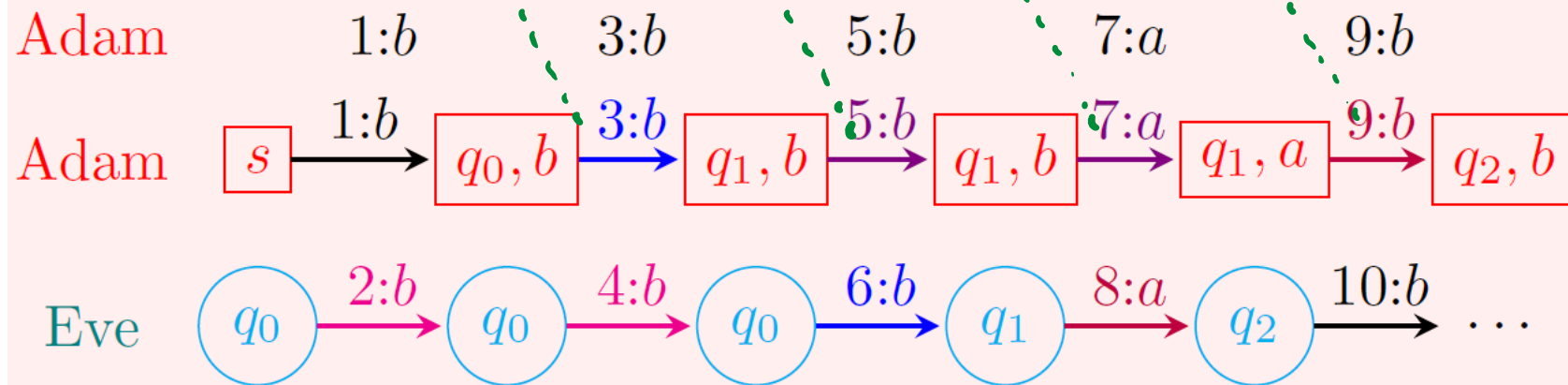


# Power of 1-Token Game

## One-Token Game



## Simulating 1-lookahead



Eve wins

1-token game



A simulates  
Delay(A)



A simulates  
 $\text{Delay}^k(A) \forall k \geq 0.$